DIFFRACTION OF BLOCH WAVE PACKETS FOR MAXWELL'S EQUATIONS

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ABSTRACT. We study, for times of order $1/h$, solutions of Maxwell's equations in an $\mathcal{O}(h^2)$ modulation of an h-periodic medium. The solutions are of slowly varying amplitude type built on Bloch plane waves with wavelength of order h. We construct accurate approximate solutions of three scale WKB type. The leading profile is both transported at the group velocity and dispersed by a Schrödinger equation given by the quadratic approximation of the Bloch dispersion relation. A weak ray average hypothesis guarantees stability. Compared to earlier work on scalar wave equations, the generator is no longer elliptic. Coercivity holds only on the complement of an infinite dimensional kernel. The system structure requires many innovations.

Key words: Geometric optics, diffractive geometric optics, Bloch waves, diffraction, electromagnetism, Maxwell's equatons.

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1. Introduction.

This paper studies the propagation of electromagnetic waves through perturbed periodic media with period $h \ll 1$.[1] We treat the resonant case where the length scale of the periodic structure is comparable to the wavelength. The observation time t satisfies $t \sim$

[1]A word on units. The Maxwell equations in vacuum have permittivities ϵ, μ and speed of light $c = 1/\sqrt{\epsilon\mu}$. Since the speed of light is $\gg 1$ in KMS or CGS units, $\epsilon\mu$ is small in those units. We perform an asymptotic analysis as $h \to 0$. Denote by Δt the unit of time. No matter what the units one has $h \ll \Delta t/\{\epsilon, \mu\}$ in this limit so there is scale separation no matter what are the values of ϵ and μ. Nevertheless it is wise to, and we choose to, work in units with $c\,\Delta t$ comparable to 1. For example centimeters for length and $\Delta t \ll 1$ equal to the number of seconds that it takes light in vacuum to traverse one centimeter. In those units $c\Delta t = 1$ and one expects that the constants in our error bounds will not be very large.

$1/h$. This is the diffractive time scale where standard Maxwell equations (without periodic structure) are approximated by Schrödinger's equation (see [12]). Wavelengths that are short compared to the period are short compared to the scale on which the coefficients vary. This is the domain of validity of standard geometric optics (see [12], [7] for the diffractive case). Wavelengths long compared to the period are analysed by standard homogenisation [8]. The interest, both mathematical and scientific, of the resonant scaling is that the speeds of propagation and diffractive effects for wave packets are given by the Bloch dispersion relation of the periodic medium and not by the symbol of the hyperbolic operator or its hyperbolic homogenisation. The propagation speeds can be radically different from those of the original equations. The new speeds must not violate the finite speed of the original equations (see Theorem 7.2 for a proof of this upper bound) but can be much smaller. This is the basis for strategies to slow light ([5], [16], [6], [29]). That in turn is one of the proposed design elements of the all optical computer. Another domain of application is photonic crystal fibers constructed with periodicity in crossection (see [27], [14], [21]). The last three examples are modeled by Maxwell's equations that are the subject of the current article.

In our papers [1], [2] we study scalar wave equations. In most cases, Maxwell's equations cannot be reduced to scalar equations. The most interesting applications require the methods of the present paper. For constant *scalar* permittivities the Maxwell equations can be reduced to the scalar wave equation. For constant ϵ, μ with ϵ a three by three matrix with three distinct positive eigenvalues, it has been known since the time of Hamilton (see [15], [11] page 610), that the characteristic polynomial is of the form $\tau^2\, Q(\tau, \xi)$ with Q an irreducible quartic polynomial. The characteristic variety is conic with nontrivial singular points. Q does not factor as the product of two quadratics in which case the variety would be the union of two (double) cones with elliptic cross-section. For variable, even scalar, permittivities, the standard derivation of second order equations by taking time derivatives works but leads to a *system* of second order equations for E coupled through lower order terms.

Maxwell's dynamic equations for unknown $E(t,x), B(t,x) \in \mathbb{R}^3 \times \mathbb{R}^3$ read

$$\partial_t \begin{pmatrix} \epsilon\, E \\ \mu\, B \end{pmatrix} \;+\; \begin{pmatrix} -\operatorname{curl} B \\ +\operatorname{curl} E \end{pmatrix} = 0 \tag{1.1}$$

whose infinitesimal generator is not elliptic. Indeed if ϵ, μ depends only on x, then $(\nabla_x \phi\,,\, \nabla_x \psi)$ is a stationary solution for any $\phi(x), \psi(x) \in C_0^\infty(\mathbb{R}^3)$. For any ball there is an infinite dimensional set of such solutions supported in the ball. This is one of the main differences of the current article with the earlier ones on scalar wave equations. The failure of ellipticity is compensated by the fact that the set of physically relevant solutions have additional strong control on their divergence. Those solutions satisfy semiclassical

ellipticity estimates (see Theorem 3.2). A second principal difference with the earlier work is that for systems it is not uncommon to have Bloch eigenvalues of multiplicity greater than one. In that case the Schrödinger equations of diffractive geometric optics are systems and we treat that possibility. A third difference with the scalar case is a simplification. For the scalar case, estimates for gradients of the error in the approximate solution were straight forward but there was a strikingly difficult argument to estimate the undifferentiated error. In the case of systems, the natural energy estimate is an L^2 estimate. Derivative estimates are proved from L^2 estimates for the time derivatives and the divergence. The remaining derivatives are estimated by an ellipticity argument. The difficult argument from the scalar case is not required.

There are two problems closely related to the ones we analyse. The first is the treatment of first order elliptic systems. The second is the treatment of the Maxwell system on the time scales of geometric optics. The second problem is solved *en passant* in §8. We have chosen to skip the first and jump directly to the Maxwell's equations. The methods that suffice for Maxwell yield the elliptic case directly.

The semiclassical estimates of the present article permit a strengthening of the earlier paper [2], where derivatives of order ≤ 1 were estimated. The new method yields estimates for derivatives of all orders.

Our three articles use corrector terms in asymptotic expansions. Article [1] correctors were constructed by an *ad hoc* method and used in test functions to study weak convergence. In [2] we introduced a a general strategy yielding accurate expansions.

The h-dependent Maxwell equations are written in the form $P^h(E^h, B^h) = 0$ with

$$P^h(t,x,\partial_t,\partial_x)(E^h,B^h) \;:=\partial_t \begin{pmatrix} \epsilon^h\, E^h \\ \mu^h\, B^h \end{pmatrix} \;+\; \begin{pmatrix} -\operatorname{curl} B^h \\ +\operatorname{curl} E^h \end{pmatrix} \;+\; M^h \begin{pmatrix} E^h \\ B^h \end{pmatrix}\,. \tag{1.2}$$

For the diffractive scaling the perturbations satisfy the following hypothesis where $\epsilon_0(x/h)$ and $\mu_0(x/h)$ are the permittivities of the unperturbed periodic structure at scale h. The 6×6 matrix valued function M^h serves for example to model dissipative effects such as Ohm's law (see §8).

Notation. *For two vectors $(e,b)\in\mathbb{C}^3\times\mathbb{C}^3$, their Hermitian inner product is denoted by $\langle e,b\rangle$ while (e,b) denotes the ordered pair in $\mathbb{C}^3\times\mathbb{C}^3$. The cross product in $\mathbb{C}^3$ is denoted by $\wedge$. The set of linear maps (homeomorphisms) on a vector space K is denoted by* $\mathrm{Hom}(K)$. *For any $\alpha=(\alpha_0,\alpha_1,\alpha_2,\alpha_3)\in\mathbb{N}^4$ the notation $\partial^\alpha_{t,x}\phi(t,x)$ means $\partial_t^{\alpha_0}\left(\prod_{i=1}^3\partial_{x_i}^{\alpha_i}\right)\phi(t,x)$.*

The Maxwell equations (1.2) are a symmetric first-order hyperbolic system for $u^h = (E^h, B^h)$

$$P^h(t,x,\partial_t,\partial_x)u^h \;=\; \partial_t(A_0^h u^h) + \sum_{j=1}^3 A_j\partial_{x_j}u^h \;+\; M^h u^h\,, \tag{1.3}$$

with symmetric matrix coefficients A_j for $j \geq 1$ defined in (8.3), (8.4) and

$$A_0^h(t,x) \ := \ \begin{pmatrix} \epsilon^h(t,x) & 0 \\ 0 & \mu^h(t,x) \end{pmatrix} .$$

Hypothesis 1.1. Diffractive time scale hypothesis. *The coefficients in* (1.2) *are given by*

$$\begin{aligned} \epsilon^h(t,x) &= \epsilon_0(x/h) + h^2 \epsilon_1(t,x,x/h) \,, \\ \mu^h(t,x) &= \mu_0(x/h) + h^2 \mu_1(t,x,x/h) \,, \\ M^h(t,x) &= h\, M(t,x,x/h) \,. \end{aligned}$$

The matrix valued functions $\epsilon_0(y)\,,\, \mu_0(y) \in C^\infty(\mathbb{T}^3)$ *are symmetric and positive definite and for all* α, $\partial^\alpha_{t,x,y}\{\epsilon_1, \mu_1, M\}(t,x,y) \in L^\infty(\mathbb{R}^{1+3} \times \mathbb{T}^3)$ *with* $\mathbb{T}^3$ *denoting the three-dimensional torus* $(\mathbb{R}/2\pi\mathbb{Z})^3$. *Define*

$$A_0^0(y) \ := \ \begin{pmatrix} \epsilon_0(y) & 0 \\ 0 & \mu_0(y) \end{pmatrix}, \qquad A_0^1(t,x,y) \ := \ \begin{pmatrix} \epsilon_1(t,x,y) & 0 \\ 0 & \mu_1(t,x,y) \end{pmatrix} .$$

Definition 1.2. *The space of* $L^2_{loc}(\mathbb{R}^3)$ *periodic functions of period* 2π *is denoted* $L^2(\mathbb{T}^3)$. *The functions* (E,B) *of* $L^2(\mathbb{T}^3)$ *with values in* $\mathbb{R}^3 \times \mathbb{R}^3$ *are normed by*

$$\int_{[0,2\pi[^3} \Big(|E|^2 + |B|^2 \Big) dx \,.$$

When normed by the equivalent expression natural in the context of Maxwell's equations

$$\int_{[0,2\pi[^3} \Big(\langle E \,,\, \epsilon_0\, E \rangle \ + \ \langle B \,,\, \mu_0\, B \rangle \Big)\ dx$$

it is denoted $L^2_{\epsilon_0,\mu_0}(\mathbb{T}^3)$.

Definition 1.3. *For* $\theta \in [0,1[^3$ *a function* $g(x)$ *on* $\mathbb{R}^3$ *is* θ**-periodic** *when* $x \to e^{-i\theta . x}\, g(x)$ *is periodic in* x *with period* 2π. *The parameter* θ *is called the* **Bloch frequency**. *The set of* $L^2_{loc}(\mathbb{R}^3)$ θ*-periodic functions is denoted* $L^2(\mathbb{T}^3_\theta)$. *With alternate norm as in Definition 1.2 it is denoted* $L^2_{\epsilon_0,\mu_0}(\mathbb{T}^3_\theta)$.

Our solutions are amplitude modulated Bloch plane waves. The plane waves are θ-periodic solutions of the unperturbed periodic Maxwell equations of the form $u = e^{\lambda t}(E(x), B(x))$. Equivalently E, B are solutions of the spectral problem

$$\lambda \begin{pmatrix} \epsilon_0(x) & 0 \\ 0 & \mu_0(x) \end{pmatrix} \begin{pmatrix} E(x) \\ B(x) \end{pmatrix} \ = \ \begin{pmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl} & 0 \end{pmatrix} \begin{pmatrix} E \\ B \end{pmatrix}, \qquad \{E\,,\,B\} \quad \theta-\mathrm{periodic}\,. \tag{1.4}$$

We recall in §6 that the spectrum at fixed θ consists of $\{0\}$ with infinite multiplicity and a discrete set of purely imaginary eigenvalues $\lambda = i\omega(\theta)$ of finite multiplicity. We label the nonzero eigenvalues according to their distance from the origin and repeat them according to their multiplicity

$$\cdots \ \leq \ \omega_{-2}(\theta) \ \leq \ \omega_{-1}(\theta) \ < \ \omega_0(\theta) = 0 \ < \ \omega_1(\theta) \ \leq \ \omega_2(\theta) \ \leq \ \cdots \,. \tag{1.5}$$

We work near an eigenvalue of constant multiplicity.

Hypothesis 1.4. *(***Constant multiplicity hypothesis.***) Fix $\underline{\theta} \neq 0$, $n \in \mathbb{Z}^*$, and denote by κ the multiplicity,*

$$\underline{\lambda} \;=\; i\omega_n(\underline{\theta}) \;\neq\; 0\,, \qquad \omega_{n-1}(\underline{\theta}) \;<\; \omega_n(\underline{\theta}) \;=\; \ldots \;=\; \omega_{n+\kappa-1}(\underline{\theta}) \;<\; \omega_{n+\kappa}(\underline{\theta})\,.$$

Assume that there are $\delta_1 > 0$, $\delta_2 > 0$ and real analytic $\omega(\theta)$ defined on $\{|\theta - \underline{\theta}| < \delta_1\}$ so that for each $|\theta - \underline{\theta}| < \delta_1$ the only point of the spectrum in $\{\lambda \,:\, |\lambda - i\omega_n(\underline{\theta})| < \delta_2\}$ is $i\omega(\theta)$ with multiplicity κ.

The hypothesis is automatically satisfied when $\kappa = 1$. An example with $\kappa > 1$ is ϵ and μ constant and scalar where every eigenvalue is of constant multiplicity two (see Example 6.6 below).

Definition 1.5. *When the constant multiplicity hypothesis 1.4 is satisfied the* **group velocity** *is defined by*

$$\mathcal{V} \;:=\; -\nabla_\theta \omega(\underline{\theta})\,.$$

Define

$$\mathbb{L}(\omega, \theta, y, \partial_y) \;:=\; i\omega \begin{pmatrix} \epsilon_0(y) & 0 \\ 0 & \mu_0(y) \end{pmatrix} - \begin{pmatrix} 0 & (i\theta + \partial_y)\wedge \\ -(i\theta + \partial_y)\wedge & 0 \end{pmatrix}. \tag{1.6}$$

$\mathbb{L}$ with domain equal to the periodic functions in $H^\infty(\mathbb{T}^3_y) := \cap_{s\geq 0} H^s(\mathbb{T}^3)$ is formally antiselfadjoint on $L^2(\mathbb{T}^3\,;\, dy)$. The method of proof of Proposition 6.3 shows that the closure has domain equal to the $v \in L^2(\mathbb{T}^3)$ so that $\mathbb{L}v \in L^2(\mathbb{T}^3)$ and that the closure is antiselfadjoint.

Definition 1.6. *Denote by Π the projection operator onto $\mathbb{K} := \ker \mathbb{L}(\omega(\underline{\theta}), \underline{\theta}, y, \partial_y)$ along the image of $\mathbb{L}$. Π is orthogonal with respect to the scalar product of $L^2(\mathbb{T}^3\,;\, dy)$ and* **not** *with respect to the scalar product of $L^2_{\epsilon_0,\mu_0}(\mathbb{T}^3)$.*

Our wave packets have group velocity $\mathcal{V}$ and travel for long times. They see the coefficients on group lines for long times. The averages of the coefficients along such long rays are important. A particular combination enters in the asymptotic description.

Definition 1.7. *For each t, x define the linear map $\gamma(t,x) \in \mathrm{Hom}(\mathbb{K})$ by*

$$\gamma(t,x) \;:=\; \big(\Pi A^0_0(t,x)\Pi\big)^{-1}\, \Pi\, (i\omega A^1_0(t,x) + M(t,x))\, \Pi\,. \tag{1.7}$$

We assume that the ray averages

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T \gamma(t, x + \mathcal{V}t)\, dt \;:=\; \widetilde{\gamma}(x)\,, \qquad \text{exist uniformly in } \; x \in \mathbb{R}^3. \tag{1.8}$$

We make a fairly weak assumption asserting that this limit is attained at an algebraic rate.

Definition 1.8. *The function γ satisfies the* **ray average hypothesis** *when* (1.8) *holds and there is a $0 \le \beta < 1$ so that for all $\alpha \in \mathbb{N} \times \mathbb{N}^3$ the solution $g_\alpha(t,x)$ of*

$$\Big(\partial_t \ + \ \mathcal{V}.\partial_x\Big) g_\alpha \ = \ \partial^\alpha_{t,x}\big(\gamma(t,x) - \widetilde{\gamma}(x - \mathcal{V}t)\big) \,, \qquad g_\alpha(0,x) \ = \ 0$$

satisfies $\langle t\rangle^{-\beta} g_\alpha \in L^\infty([0,\infty[\times\mathbb{R}^3)$ where $\langle t\rangle := (1+t^2)^{1/2}$.

This hypothesis, introduced in [2], is discussed in §9.1. Our main theorem gives an approximate solution and an error estimate. In the theorem, $\mathcal{T}$ is a new variable, a slow time. In the approximate solution it is replaced by ht. In addition there is a $\mathbb{K}$ valued function, $\widetilde{w}_0(\mathcal{T},x)$. Abusing notation, the value at $(\mathcal{T},x)$ is a function of y denoted $\widetilde{w}_0(\mathcal{T},x,y)$.

Theorem 1.9. *Assume that γ satisfies the ray average hypothesis with parameter $0 \le \beta < 1$. For $f \in \mathcal{S}(\mathbb{R}^3;\mathbb{K})$ define $w_0 := \widetilde{w}_0(\mathcal{T}, x - \mathcal{V}t)$, where $\widetilde{w}_0 \in C^\infty(\mathbb{R}\,; \mathcal{S}(\mathbb{R}^3;\mathbb{K}))$ is the unique solution of the initial value problem for Schrödinger's equation*

$$\Big(\partial_{\mathcal{T}} + \frac{1}{2} i\ \partial^2_\theta \omega(\partial_x, \partial_x) + \widetilde{\gamma}(x)\Big)\widetilde{w}_0 \ = \ 0, \qquad \widetilde{w}_0(0,x) = f(x). \tag{1.9}$$

Define a family of approximate solutions

$$\underline{v}^h(t,x) \ := \ e^{i(\omega(\underline{\theta})t + \underline{\theta}.x)/h}\ w_0(ht,t,x,x/h)$$

and let $\underline{u}^h$ denote the exact solution of $P^h \underline{u}^h = 0$ with $\underline{u}^h|_{t=0} = \underline{v}^h|_{t=0}$. Then

$$\forall\, T > 0,\ \alpha \in \mathbb{N}^4,\ \exists\, C(\alpha,T),\ \forall\, h \in]0,1[, \quad \sup_{t\in[0,T/h]} \big\|(x\,,\,h\,\partial_{t,x})^\alpha(\underline{u}^h - \underline{v}^h)\big\|_{L^2(\mathbb{R}^3)} \ \le \ C\,h^{1-\beta}\,,$$

with $0 \le \beta < 1$ from the ray average hypothesis of Definition 1.8.

Remark 1.10. i. *The operator $\Pi A^0_0 \Pi$ is positive definite on $\mathbb{K}$. When $M = 0$, mulitplying the Schrödinger equation (1.9) by $\Pi A^0_0 \Pi$ shows that $\int \langle \Pi A^0_0 \Pi \widetilde{w}_0\,,\,\widetilde{w}_0\rangle\,dx$ is conserved.* **ii.** *The principal part of equation* (1.9) *is scalar. Coupling occurs through $\widetilde{\gamma}$ (see Remark 9.3).*

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2. $L^2(\mathbb{R}^3)$ ESTIMATES

Suppose given real symmetric permittivities

$$\epsilon^h(t,x)\,,\ \mu^h(t,x)\ \in\ C^\infty\big(\mathbb{R}^{1+3}\times]0,1[_h\,;\,\mathrm{Hom}(\mathbb{R}^3)\big)$$

satisfying the positivity constraint

$$\forall\, t,x,h \qquad 0\ <\ cI\ \le\ \epsilon^h(t,x)\,,\ \mu^h(t,x)\ \le\ CI\,. \tag{2.1}$$

Consider the dynamic Maxwell equations in a medium that varies on scale $0<h<<1$

$$\partial_t(\epsilon^h(t,x)\,E^h)\ =\ \mathrm{curl}\,B^h, \qquad \partial_t(\mu^h(t,x)B^h)\ =\ -\,\mathrm{curl}\,E^h\,. \tag{2.2}$$

This is a symmetric hyperbolic system. The energy identity for solutions is

$$\partial_t\int_{\mathbb{R}^3}\langle E^h,\epsilon^h E^h\rangle+\langle B^h,\mu^h B^h\rangle\ dx\ =\ -\int_{\mathbb{R}^3}\langle E^h,\partial_t\epsilon^h E^h\rangle\ +\ \langle B^h,\partial_t\mu^h B^h\rangle\ dx\,. \tag{2.3}$$

Large time derivatives of ϵ^h,μ^h can lead to rapid growth of energy. On the other hand if $\partial_t\epsilon^h,\partial_t\mu^h$ are bounded (resp. $\mathcal{O}(h)$) one has uniform estimates for the L^2 norm for $t=\mathcal{O}(1)$ (resp. $t=\mathcal{O}(1/h)$). Corresponding estimates for derivatives is subtle because the coefficients are rapidly varying in space. Our strategy is to derive estimates for time derivatives and for $\mathrm{div}E^h,\mathrm{div}B^h$. Then estimate spatial derivatives using an elliptic estimate from the next section.

3. COERCIVITY

The generator of the dynamic equations (1.2) is not elliptic. However, the special structure of Maxwell's equations yields supplementary bounds on $\mathrm{div}E$ and $\mathrm{div}B$. The over determined system consisting of the generator together with divergence is elliptic. For problems with coefficients oscillating in x on scale h, estimates in semiclassical Sobolev spaces are natural.

Definition 3.1. *The semiclassical Sobolev norm $H^m(\mathbb{R}^3)$ denoted $\|\cdot\|_{H^m_h(\mathbb{R}^3)}$ is defined by*

$$\Big(\sum_{|\alpha|\le m}\int\big|(h\partial_x)^\alpha u\big|^2\,dx\Big)^{1/2}.$$

For a function $u(t,x)$, integer $m\ge 0$ and $h\in]0,1[$ define the semiclassical norm with derivatives in space and time,

$$\big\|u(t)\big\|^2_{m,h}\ :=\ \sum_{|\alpha|\le m}\ \big\|(h\,\partial_{t,x})^\alpha u(t)\big\|^2_{L^2(\mathbb{R}^3)}\,. \tag{3.1}$$

Denote by $C^k_h(\mathbb{R}^{1+3})$ the set of families $\{w^h\,:\,0<h<1\}\subset C^k(\mathbb{R}^{1+3})$ so that

$$\sup_{0<h<1}\ \sum_{|\alpha|\le k}\ \big\|(h\,\partial_{t,x})^\alpha w^h\big\|_{L^\infty(\mathbb{R}^{1+3})}\ <\ \infty. \tag{3.2}$$

The same notation is used for functions valued in any finite dimensional real or complex vector space. The left hand side of (3.2) *serves as norm making* C^k_h *a Banach space. The Fréchet space* C^∞_h *is defined as* $\cap_k C^k_h$.

Theorem 3.2. *For* $\epsilon^h(t,x), \mu^h(t,x) \in C^\infty_h(\mathbb{R}^{1+3})$ *and* $m \in \mathbb{N}$ *there is a constant* $C(m)$ *so that for all* $E, B \in H^m(\mathbb{R}^3)$, $t \in \mathbb{R}$, *and* $h > 0$

$$\big\|E(t)\big\|_{H^m_h(\mathbb{R}^3)} \leq C(m)\left(\big\|h\,\mathrm{curl}\,E(t)\big\|_{H^{m-1}_h(\mathbb{R}^3)} + \big\|h\,\mathrm{div}(\epsilon^h(t,x)E(t))\big\|_{H^{m-1}_h(\mathbb{R}^3)} + \big\|E(t)\big\|_{H^{m-1}_h(\mathbb{R}^3)}\right),$$

$$\big\|B(t)\big\|_{H^m_h(\mathbb{R}^3)} \leq C(m)\left(\big\|h\,\mathrm{curl}\,B(t)\big\|_{H^{m-1}_h(\mathbb{R}^3)} + \big\|h\,\mathrm{div}(\mu^h(t,x)B(t))\big\|_{H^{m-1}_h(\mathbb{R}^3)} + \big\|B(t)\big\|_{H^{m-1}_h(\mathbb{R}^3)}\right).$$

Proof. The inequalities for E and B are identical so it suffices to consider E. The variable t is simply a parameter so it suffices to consider t fixed. The key estimate is the following.

Lemma 3.3. *If* $\underline{\epsilon}(x)$ *is a symmetric matrix valued function so that* $\underline{\epsilon} \geq cI > 0$ *for all* x, *and* $\partial^\alpha_x \underline{\epsilon} \in L^\infty(\mathbb{R}^3)$ *for all* $\alpha \in \mathbb{N}^3$, *then for each* $m \in \mathbb{N}$ *there is a constant* $C(m)$ *so that for all* $E \in H^m(\mathbb{R}^3)$

$$\big\|E\big\|_{H^m(\mathbb{R}^3)} \;\leq\; C(m)\left(\big\|\mathrm{curl}\,E\big\|_{H^{m-1}(\mathbb{R}^3)} + \big\|\mathrm{div}(\underline{\epsilon}(x)E)\big\|_{H^{m-1}(\mathbb{R}^3)} + \big\|E\big\|_{H^{m-1}(\mathbb{R}^3)}\right). \tag{3.3}$$

Proof. The integrand in

$$\int_{\mathbb{R}^3} |\mathrm{div}(\underline{\epsilon}(x)\,E)|^2 \;+\; |\mathrm{curl}\,E|^2\;dx$$

is a quadratic form in E and its first derivatives. The terms quadratic in the derivatives of E have the form

$$\sum_{1\leq i,j\leq 3} \big\langle a_{i,j}(x)\,\partial_i E\,,\,\partial_j E\big\rangle$$

with uniquely determined real matrix valued functions $a_{i,j}(x)$ with $a_{j,i}$ equal to the transpose of $a_{i,j}$. Introduce the the symbol $\sum_{i,j} a_{i,j}(x)\xi_i\xi_j$. The definition implies that for any x, $\mathbf{e}$ and ξ in $\mathbb{R}^3$

$$\sum_{i,j}\big\langle a_{i,j}(x)\xi_i\xi_j\mathbf{e}\,,\,\mathbf{e}\big\rangle \;=\; |\langle \xi\,,\,\underline{\epsilon}(x)\mathbf{e}\rangle|^2 \;+\; |\xi\wedge\mathbf{e}|^2\,. \tag{3.4}$$

Choose $0 < c \leq 1$ so that for all x, $\underline{\epsilon}(x) \geq cI$. The lemma is a consequence of Gärding's inequality once we prove that for all real $x, \mathbf{e}, \xi$

$$|\langle \xi\,,\,\underline{\epsilon}(x)\mathbf{e}\rangle|^2 \;+\; |\xi\wedge\mathbf{e}|^2 \;\geq\; c\,|\xi|^2\,|\mathbf{e}|^2/2\,. \tag{3.5}$$

By homogeneity it suffices to consider $|\xi| = 1$.

Decompose $\mathbf{e} = \mathbf{e}_\parallel + \mathbf{e}_\perp$ into parts parallel and perpendicular to ξ. The definition of c yields

$$|\mathbf{e}_\perp|^2 \leq |\mathbf{e}|^2/2 \quad\Longrightarrow\quad |\langle \xi\,,\,\underline{\epsilon}(x)\mathbf{e}_\parallel\rangle|^2 \;\geq\; c\,|\mathbf{e}_\parallel|^2 \;=\; c\big(|\mathbf{e}|^2 - |\mathbf{e}_\perp|^2\big) \;\geq\; c\,|\mathbf{e}|^2/2\,. \tag{3.6}$$

On the other hand since $c \leq 1$

$$|\mathbf{e}_\perp|^2 \geq |\mathbf{e}|^2/2 \quad \Longrightarrow \quad |\xi \wedge \mathbf{e}|^2 \;=\; |\mathbf{e}_\perp|^2 \;\geq\; |\mathbf{e}|^2/2 \;\geq\; c\,|\mathbf{e}|^2/2\,. \tag{3.7}$$

Estimates (3.6) and (3.7) imply (3.5) completing the proof of the lemma. □

Scaling shows that (3.3) implies the h dependent coercivity of Theorem 3.2. □

4. Stability

Theorem 4.1. *Let P^h, be the operator (1.2) with $\epsilon^h, \partial_t\epsilon^h, \mu^h, \partial_t\mu^h, M^h \in C_h^\infty(\mathbb{R}^{1+3})$. If $f, g \in H^\infty(\mathbb{R}^3) := \cap_s H^s(\mathbb{R}^3)$ then there is a unique family of solutions $u^h = (E^h, B^h) \in C^\infty(\mathbb{R}\,;\, H^\infty(\mathbb{R}^3))$ to $P^h u^h = 0$, with $u^h(0) = (f, g)$. For each $m \in \mathbb{N}$ there is a constant $c(m)$ so that with*

$$C(m,h) \;:=\; \sum_{|\alpha|\leq m} \big\|(h\partial_{t,x})^\alpha(\partial_t\epsilon^h, \partial_t\mu^h, M^h)\big\|_{L^\infty(\mathbb{R}^{1+3})}$$

one has

$$\big\|u^h(t)\big\|_{m,h} \;\leq\; c(m)\, e^{c(m)\,C(m,h)\,t}\; \big\|u^h(0)\big\|_{m,h}\,. \tag{4.1}$$

Remark 4.2. *If $M^h = 0$, one has $\partial_t\big(\mathrm{div}\big(\epsilon^h(t,x)\,E^h\big)\big) = 0$ and $\partial_t\big(\mathrm{div}\big(\mu^h(t,x)\,B^h\big)\big) = 0$. In particular one has*

$$\mathrm{div}\big(\epsilon^h(t,x)\,E^h\big) \;=\; 0, \qquad \mathrm{div}\big(\mu^h(t,x)\,B^h\big) \;=\; 0\,, \tag{4.2}$$

as soon as these identities hold at $t = 0$.

Remark 4.3. *For the analysis on the diffractive scale, the theorem is applied with $C(m,h) = \mathcal{O}(h)$ as $h \to 0$. In that case one has uniform bounds for $t = \mathcal{O}(1/h)$.*

Remark 4.4. *This careful accounting of derivatives in Theorem 4.1 is at the heart of extenting the results of this paper to equations whose coefficients are only finitely differentiable.*

Proof. The existence for fixed h is classical. The estimate for $m = 0$ follows from Gronwall's inequality together with

$$\begin{aligned}
\partial_t \int_{\mathbb{R}^3} \big(\langle \epsilon^h\, E^h\,,\, E^h\rangle + \langle \mu^h\, B^h\,,\, B^h\rangle\big)\, dx &= -\int_{\mathbb{R}^3} \big(\langle \partial_t\epsilon^h\, E^h\,,\, E^h\rangle + \langle \partial_t\mu^h\, B^h\,,\, B^h\rangle\big)\, dx \\
&\qquad + 2\int_{\mathbb{R}^3} \langle u^h, M^h u^h\rangle\, dx \\
&\leq C\Big(\|\partial_t\epsilon^h(t), \partial_t\mu^h(t)\|_{L^\infty(\mathbb{R}^3)} \;+\; \|M^h(t)\|_{L^\infty(\mathbb{R}^3)}\Big)\|(E^h, B^h)(t)\|_{L^2(\mathbb{R}^3)}\,.
\end{aligned}$$

The proof for arbitrary m is by induction. Assume the estimate proved for m. Denote by $K^h(t,s)$ the evolution operator associated to the equation $P^h w = 0$. Precisely, for a distribution $g \in \mathcal{D}'(\mathbb{R}^3)$, $w(t) = K^h(t,s)g$ is the unique solution of the Cauchy problem

$$P^h\, w \;=\; 0\,, \qquad w\big|_{t=s} \;=\; g\,.$$

The inductive hypothesis is a bound

$$\|K^h(t,s)g\|_{m,h} \;\le\; c(m)\, e^{c(m)C(m,h)|t-s|}\, \|g\|_{H^m_h(\mathbb{R}^3)}\,.$$

The Duhamel relation

$$v^h(t) \;=\; K^h(t,0)v^h(0) \;+\; \int_0^t K^h(t,s)\; P^h(v^h)(s)\; ds$$

then yields the estimate

$$\|v^h(t)\|_{m,h} \le c(m)\Big(e^{c(m)C(m,h)t}\|v^h(0)\|_{m,h} + \int_0^t e^{c(m)C(m,h)(t-s)}\; \|P^h(v^h)(s)\|_{m,h}\; ds\Big).$$

The key point is that one cannot simply apply the estimate at level m to $v^h := h\partial u^h$. Doing so yields

$$\|h\partial u^h(t)\|_{m,h} \le c(m)\Big(e^{c(m)C(m,h)t}\|h\partial u^h(0)\|_{m,h} + \int_0^t e^{c(m)C(m,h)(t-s)}\; \|P^h(h\partial u^h)(s)\|_{m,h}\; ds\Big).$$

Write $P^h(h\partial) = (h\partial)P^h + [P^h, h\partial]$. The first term vanishes when applied to u. The commutator $[P^h, h\partial]$ is

$$h\partial_t \begin{pmatrix} \partial\epsilon^h & 0 \\ 0 & \partial\mu^h \end{pmatrix} \;+\; h\,\partial M^h\,.$$

If ∂ were a derivative with respect to x then $\partial_x\{\epsilon^h, \mu^h\} \sim 1/h$ leading to an unacceptably large contribution. Instead of estimating all derivatives we estimate only ∂_t, $\mathrm{div}\{\epsilon^h E^h, \mu^h B^h\}$, and curl. The time derivatives of ϵ^h, μ^h are bounded. The divergence and curl play a special role in Maxwell's equations. The remaining spatial derivatives are recovered using coercivity. [2]

For ∂_t compute

$$[P^h, h\partial_t] \;=\; \begin{pmatrix} \partial_t\epsilon^h & 0 \\ 0 & \partial_t\mu^h \end{pmatrix} h\partial_t \;+\; h\,\partial_t M^h\,.$$

Estimate

$$\|P^h(h\partial_t u^h)(s)\|_{m,h} \;\le\; C(m+1,h)\, \|u^h(s)\|_{m+1,h}\,.$$

The Duhamel estimate yields

(4.3)

$$\|h\partial_t u^h(t)\|_{m,h} \le c(m)\Big(e^{c(m)C(m,h)t}\|u^h(0)\|_{m+1,h} + \int_0^t e^{c(m)C(m,h)(t-s)}\; C(m+1,h)\|u^h(s)\|_{m+1,h}\, ds\Big).$$

[2] The use of $\partial_t, \mathrm{div}, \mathrm{curl}$ is reminiscent of the use of $\partial_t + \mathbf{v}\partial_x, \mathrm{div}, \mathrm{curl}$, and tangential derivatives for the inviscid compressible Euler equations in [24].

Write the Maxwell equations as

$$\big(h\,\mathrm{curl}\,B^h\,,\;-h\,\mathrm{curl}\,E^h\big)\;=\;h\partial_t(\epsilon^h\,E^h\,,\;\mu^h\,B^h)+h\,M^h\,u^h\,.$$

Therefore

$$\big\|h\,\mathrm{curl}\,E^h(t)\,,\;h\,\mathrm{curl}\,B^h(t)\big\|_{m,h}\;\le\;\Big(\text{r.h.s. of (4.3)}+\;\|h\,u^h(t)\|_{m,h}\Big)C(m,h)\,.$$

Use the fundamental theorem of calculus to estimate

$$\|h\,u^h(t)\;\|_{m,h}\le\|h\,u^h(0)\;\|_{m,h}+\int_0^t\|h\partial_t u^h(s)\|_{m,h}\;ds\,.$$

Wasting a derivative in the first term yields

$$\le\|u^h(0)\;\|_{m+1,h}+\int_0^t\|u^h(s)\|_{m+1,h}\;ds.$$

Combining the last four assertions yields

$$\big\|h\,\mathrm{curl}\,E^h(t)\,,\;h\,\mathrm{curl}\,B^h(t)\big\|_{m,h}\;\le\;\text{r.h.s. of (4.3)}\,. \tag{4.4}$$

Next estimate $\|h\,\mathrm{div}\,\epsilon^h E^h, h\,\mathrm{div}\,\mu^h B^h\|_{m,h}$. Write M^h in block form with 3×3 blocks

$$M^h\;=\;\begin{pmatrix}M^h_{11}&M^h_{12}\\M^h_{21}&M^h_{22}\end{pmatrix}\,.$$

Therefore

$$\partial_t\big(\mathrm{div}(\epsilon^h E^h)\big)\;=\;\mathrm{div}(\epsilon^h E^h)_t\;=\;-\mathrm{div}(M^h_{11}E^h+M^h_{12}B^h)\,.$$

Integrate to find

$$\begin{aligned}\|h\,\mathrm{div}(\epsilon^h E^h)(t)\|_{m,h}\;&\le\;\|h\,\mathrm{div}(\epsilon^h E^h)(0)\|_{m,h}\;+\;\int_0^t\|M^h_{11}E^h(s)+M^h_{12}B^h(s)\|_{m,h}\;ds\\&\le\;C(m,h)\,\|u^h(0)\|_{m+1,h}\;+\;\int_0^t C(m,h)\|u^h(s)\|_{m,h}\;ds\,.\end{aligned}$$

Performing the analogous estimate for $h\,\mathrm{div}(\mu^h B^h)$ and replacing $\|\;\|_{m,h}$ on the right by the larger $\|\;\|_{m+1,h}$ yields

$$\|h\,\mathrm{div}(\epsilon^h E^h)(t)\,,\;h\,\mathrm{div}(\mu^h B^h)\|_{m,h}\;\le\;\text{r.h.s. of (4.3)}\,. \tag{4.5}$$

Combining (4.3), (4.4) and (4.5), the inductive hypothesis, and the coercivity estimate from Theorem 3.2 yields

$$\|u^h(t)\|_{m+1,h}\le c(m)\Big(\|u^h(0)\|_{m+1,h}+\int_0^t e^{c(m)C(m,h)(t-s)}\;C(m+1,h)\|u^h(s)\|_{m+1,h}\,ds\Big). \tag{4.6}$$

The theorem follows from Gronwall's inequality. $\square$

5. Stationary solutions

When ϵ and μ depend only on x, there is a conserved L^2 norm for the solution of (1.1),

$$\partial_t \int_{\mathbb{R}^3} \langle E\,,\, \epsilon(x)\, E\rangle \;+\; \langle B\,,\, \mu(x)\, B\rangle\; dx \;=\; 0\,. \tag{5.1}$$

Equations (1.1) have an infinite dimensional space of stationary solutions. The set of functions satisfying $\operatorname{div}\epsilon(x)E = \operatorname{div}\mu(x)B = 0$ is invariant and orthogonal in the conserved L^2 scalar product to the stationary solutions.

To prove this assertion we use the Fourier Transform. For any $u \in L^2(\mathbb{R}^3)$, we have

$$\hat{u}(\xi) \;=\; (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{-ix.\xi}\; u(x)\; dx, \qquad u(x) \;=\; (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{ix.\xi}\; \hat{u}(\xi)\; d\xi\,.$$

The Fourier Transform is unitary on $L^2(\mathbb{R}^3, dx)$.

Theorem 5.1. *When $\epsilon(x), \mu(x)$ depend only on x, $u(x) = (E(x), B(x)) \in L^2(\mathbb{R}^3)$ is a stationary solution of the dynamic Maxwell equations* (1.1) *if and only if* $\operatorname{curl} E = \operatorname{curl} B = 0$*. The orthogonal complement of these data in $L^2_{\epsilon,\mu}(\mathbb{R}^3)$ normed by $\int \langle \epsilon E, E\rangle + \langle \mu B, B\rangle\, dx$ is invariant under the flow and consists exactly of the solutions satisfying* (4.2)*. The fields $(\operatorname{grad}\phi, \operatorname{grad}\psi)$ with $\phi, \psi \in H^1(\mathbb{R}^3)$ are $L^2(\mathbb{R}^3)$-dense in the stationary solutions.*

Proof. The first assertion is obvious.

The orthogonal complement of the stationary solutions is invariant because the evolution is unitary.

Next prove the density of gradients. The field E satisfies $\operatorname{curl} E = 0$ if and only if $\xi \wedge \widehat{E}(\xi) = 0$, that is the Fourier Transform $\widehat{E}(\xi)$ is parallel to ξ for almost all ξ. For $\xi \neq 0$ $\widehat{E} = \xi f(\xi)$ uniquely defines the scalar valued $f(\xi)$. Choose a smooth cutoff function $0 \le \chi \le 1$ vanishing on a neighborhood of $\xi = 0$ and identically equal to 1 outside a compact set. Then E is the L^2-limit as n tends to infinity of the field with Fourier transform equal to $\chi(n\xi)\, \xi\, f(\xi)$. Define $\widehat{\phi^n} = \chi(n\xi) f$ so $\phi \in H^1(\mathbb{R}^3)$ and $\operatorname{grad}\phi^n \to E$ in L^2.

Using the density of the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ in $H^1(\mathbb{R}^3)$, shows that a vector E is orthogonal to the stationary states if and only if

$$\int_{\mathbb{R}^3} \langle \epsilon E, \operatorname{grad}\phi\rangle\; dx \;=\; 0, \quad \text{for all} \quad \phi \in \mathcal{S}(\mathbb{R}^3)\,.$$

This is the definition of $\operatorname{div}(\epsilon E) = 0$ in the sense of tempered distributions. □

6. The theory of Floquet and Bloch

6.1. **Bloch Transform.** We recall the essentials of the method of Floquet and Bloch (see for example [13], [9], [10], [26], [30], [8]).

Write each $\xi \in \mathbb{R}^3$ as $\mathbf{n}+\theta$ with uniquely determined $\mathbf{n} \in \mathbb{Z}^3$ and $\theta \in [0,1[^3$. Expressing $u(x)$ in terms of its Fourier transform, $\hat{u}(\xi)$, yields

$$(6.1) \qquad u(x) \;=\; (2\pi)^{-3/2} \int_{[0,1[^3} e^{i\theta.x} \Big(\sum_{\mathbf{n}\in\mathbb{Z}^3} e^{i\mathbf{n}.x} \hat{u}(\theta+\mathbf{n}) \Big)\, d\theta\,.$$

The parentheses enclose the Fourier series expansion of a function periodic in x with period 2π. Considered as a function of x, the integrand is θ-periodic in the sense of Definition 1.3. Identity (6.1) decomposes $L^2(\mathbb{R}^3)$ as the direct integral over θ of the Hilbert spaces of θ-periodic functions belonging to $L^2_{\rm loc}(\mathbb{R}^3)$.

Proposition 6.1. *The map that associates to u its Bloch wave expansion u_θ*

$$(6.2) \qquad L^2(\mathbb{R}^3) \ni u \quad \mapsto \quad u_\theta(x) \;:=\; (2\pi)^{-3/2}\, e^{i\theta.x} \Big(\sum_{\mathbf{n}\in\mathbb{Z}^3} e^{i\mathbf{n}.x} \hat{u}(\theta+\mathbf{n}) \Big) \;\in\; L^2(\mathbb{T}^3_\theta)$$

yields a unitary decomposition of $L^2(\mathbb{R}^3)$ as the direct integral over $\theta \in [0,1[^3$ of $L^2(\mathbb{T}^3_\theta)$. The inverse is given by

$$u(x) \;=\; \int_{[0,1[^3} u_\theta(x)\; d\theta \qquad \text{with} \qquad \|u\|^2_{L^2(\mathbb{R}^3)} \;=\; \int_{[0,1[^3} \|u_\theta\|^2_{L^2(\mathbb{T}^3)}\; d\theta\,.$$

The map $u \mapsto e^{-i\theta.x} u_\theta$ is a unitary map $L^2(\mathbb{R}^3) \to L^2([0,1[^3\,;\, L^2(\mathbb{T}^3))$.

Remark 6.2. *The partial derivatives and 2π-translates of θ-periodic functions are θ-periodic and the product of a θ-periodic function by a 2π-periodic function is θ-periodic. Therefore partial differential operators with 2π-periodic coefficients map θ-periodic functions to themselves. The Bloch decomposition reduces these operators.*

6.2. **Maxwell's equations.** The method of Floquet-Bloch applies to Maxwell's equations (see for example [8] and [28]). The delicate part for us is the infinite dimensional kernel and the degeneration of the coercivity estimates as $h \to 0$. From here to the end of this section the Bloch strategy is used to analyse Maxwell's equations

$$(6.3) \qquad \begin{pmatrix} \epsilon_0(x) & 0 \\ 0 & \mu_0(x) \end{pmatrix} \partial_t \;-\; G\,, \qquad G \;:=\; \begin{pmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl} & 0 \end{pmatrix}$$

in the case of periodic $\epsilon(x)$ and $\mu(x)$. It suffices to analyse its action as a map on $L^2(\mathbb{T}^3_\theta)$

Proposition 6.3. *If A_j are symmetric matrices then the operator $L = \sum_j A_j \partial_j$ satisfies*

$$(6.4) \qquad \forall u, v \in L^2(\mathbb{T}^3_\theta) \cap C^\infty \qquad \langle Lu, v\rangle \;=\; -\langle u, Lv\rangle\,.$$

Denote by $\overline{L}$ the closure of the operator so defined and by L^* the Hilbert space adjoint. Then $\overline{L}$ is antiselfadjoint with domain equal to the set of $u \in L^2(\mathbb{T}^3_\theta)$ so that $\sum_j A_j \partial_j u \in L^2(\mathbb{T}^3_\theta)$ in the sense of distributions.

Proof. To prove (6.4), write $u = e^{i\theta.x}\tilde{u}$ and similarly v with periodic $\tilde{u}, \tilde{v}$. Then,

$$\begin{aligned}
\langle Lu, v\rangle &= \int_{[0,2\pi[^3} \langle A_j\partial_j u\,,\, v\rangle\, dx \;=\; \int_{[0,2\pi[^3} \langle A_j\partial_j(e^{i\theta.x}\tilde{u})\,,\, e^{i\theta.x}\tilde{v}\rangle\, dx \\
&= \int_{[0,2\pi[^3} \langle e^{i\theta.x}A_j(\partial_j + i\theta_j)\tilde{u}\,,\, e^{i\theta.x}\tilde{v}\rangle\, dx \;=\; \int_{[0,2\pi[^3} \langle A_j(\partial_j + i\theta_j)\tilde{u}\,,\, \tilde{v}\rangle\, dx \\
&= -\int_{[0,2\pi[^3} \langle \tilde{u}\,,\, A_j(\partial_j + i\theta_j)\tilde{v}\rangle\, dx\,,
\end{aligned}$$

the last step by integration by parts and periodic boundary conditions.

Identity (6.4) implies that $L^* \supset -L$ in the sense that the left hand side is an extension of the right. The definition of distribution derivative implies that $L^*u = f \in L^2(\mathbb{T}^3_\theta)$ if and only if $-(\sum_j A_j\partial_j)u = f$ in the sense of distributions.

In that case denote by J_δ a standard mollifier. Since θ-periodic functions are invariant under translations, $J_\delta u \in C^\infty \cap L^2(\mathbb{T}^3_\theta)$. Since J_δ commutes with $A_j\partial_j$ so $u^\delta := J_\delta u$ satisfies $-Lu^\delta = J^\delta f$. Passing to the limit shows that u belongs to the domain of $\overline{L}$ and $\overline{L}u = -f$. Thus $L^* \subset -\overline{L}$. □

The spaces $L^2(\mathbb{T}^3_\theta)$ depend on θ. The following proposition allows one to apply standard results in perturbation theory.

Proposition 6.4. *The unitary map $L^2(\mathbb{T}^3) \ni v \mapsto e^{i\,\theta.x}\, v \in L^2(\mathbb{T}^3_\theta)$ intertwines the operators*

$$\begin{pmatrix} 0 & \partial_x\wedge \\ -\partial_x\wedge & 0 \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} 0 & (\partial_x + i\theta)\wedge \\ -(\partial_x + i\theta)\wedge & 0 \end{pmatrix}.$$

The former acts on $L^2(\mathbb{T}^3_\theta)$ and the latter on the θ independent space $L^2(\mathbb{T}^3)$. The latter family of operators depends analytically on θ.

Proof. The unitary map commutes with multiplication operators and intertwines the antiselfadjoint operators ∂_j on $L^2(\mathbb{T}^3_\theta)$ and $\partial_j + i\theta_j$ on $L^2(\mathbb{T}^3)$. This yields the desired result. □

The straight forward proof of the next result is omitted.

Proposition 6.5. *A function $u(x) = (E(x), B(x)) \in L^2(\mathbb{R}^3)$ is a stationary solution of* (1.1) *if and only if its Bloch wave expansion $(E_\theta, B_\theta) \in L^2(\mathbb{T}^3_\theta)$ satisfies $G\,(E_\theta, B_\theta) = 0$ for almost all $\theta \in [0,1[^3$ with G from* (6.3). *A function u is in the invariant space of functions orthogonal in $L^2_{\epsilon_0,\mu_0}(\mathbb{T}^3_\theta)$ to these stationary solutions if and only if its expansion satisfies* $\mathrm{div}(\epsilon_0 E_\theta) = \mathrm{div}(\mu_0 B_\theta) = 0$ *in the sense of distributions.*

6.3. **Bloch spectral theory.** Consider fixed periodic $\epsilon_0(x), \mu_0(x) \in C^\infty(\mathbb{T}^3)$. The function

$$u(t,x) \ := \ e^{\lambda t}\ \big(E(x)\,,\,B(x)\big)$$

satisfies Maxwell's equations (1.1) if and only if

$$\lambda \begin{pmatrix} \epsilon_0(x) & 0 \\ 0 & \mu_0(x) \end{pmatrix} \begin{pmatrix} E(x) \\ B(x) \end{pmatrix} \ = \ \begin{pmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl} & 0 \end{pmatrix} \begin{pmatrix} E \\ B \end{pmatrix}. \tag{6.5}$$

In the same way a θ-periodic $u = e^{\lambda t}(E(x), B(x))$ is a θ-periodic solution of Maxwell's equations if and only if E, B is a solution of (6.5) and $(E\,,\,B)$ is θ-periodic. The function u is then a solution of (2.2).

When each of ϵ_0 and μ_0 is a positive constant times the identity the change of variable $\widetilde{E}, \widetilde{B} = \epsilon_0^{1/2}E, \mu_0^{1/2}B$ reduces to the case $\epsilon_0 = \mu_0 = I$. For that case the eigenvalue problem is exactly solved in the next example.

Example 6.6. *In case $\epsilon_0 = \mu_0 = I$ the problem is translation invariant. Denote by T_ℓ the translation operator $u(x) \mapsto u(x - \ell)$ acting on $L^2(\mathbb{T}^3_\theta)$. The antiselfadjoint G commutes with T_ℓ so the eigenspaces of T_ℓ are invariant by G.*

For given $k \in \mathbb{Z}^3$ denote by E_k the subspace consisting of exponentials $e^{ik.x}e^{i\theta.x}(\mathbf{e}, \mathbf{b})$ when $\mathbf{e}, \mathbf{b}$ run in $\mathbb{R}^3$. E_k consists of eigenvectors of T_ℓ with eigenvalue $e^{i(k+\theta).\ell}$. Choose the vector ℓ so that the $\ell_j/2\pi$ are rationally independent. Then the eigenvalues for distinct k are distinct, so the spectral decomposition of T_ℓ is $L^2(\mathbb{T}^3_\theta) = \oplus_\perp E_k$.

It follows that $G(E_k) \subset E_k$ for all $k \in \mathbb{Z}^3$. It suffices to diagonalize the restriction of G to each E_k. Compute

$$G\left(e^{i\theta.x}e^{ik.x}\begin{pmatrix} \mathbf{e} \\ \mathbf{b} \end{pmatrix}\right) \ = \ \begin{pmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl} & 0 \end{pmatrix} \left(e^{i\theta.x}\begin{pmatrix} e^{ik.x}\mathbf{e} \\ e^{ik.x}\mathbf{b} \end{pmatrix}\right) \ = \ e^{i\theta.x}e^{ik.x}\begin{pmatrix} i(k+\theta)\wedge\mathbf{b} \\ -i(k+\theta)\wedge\mathbf{e} \end{pmatrix}.$$

To have eigenvalue $\lambda = i\omega$ it is necessary and sufficient that ω is an eigenvalue of $G_0 \in$ $\mathrm{Hom}(\mathbb{C}^6)$

$$G_0 \begin{pmatrix} \mathbf{e} \\ \mathbf{b} \end{pmatrix} \ := \ \begin{pmatrix} (k+\theta)\wedge\mathbf{b} \\ -(k+\theta)\wedge\mathbf{e} \end{pmatrix} \ = \ \omega \begin{pmatrix} \mathbf{e} \\ \mathbf{b} \end{pmatrix}, \tag{6.6}$$

One has eigenvalue 0 if and only if both $\mathbf{e}$ and $\mathbf{b}$ are parallel to $k + \theta$ This kernel has dimension equal to two.

The orthogonal space has dimension 4 and consists of vectors with both $\mathbf{e}$ and $\mathbf{b}$ orthogonal to $k+\theta$. Since $(k+\theta) \perp \mathbf{b}$, $(k+\theta)\wedge(k+\theta)\wedge\mathbf{b} = -|k+\theta|^2\mathbf{b}$. Using (6.6) compute

$$-|k+\theta|^2\mathbf{b} = (k+\theta)\wedge(k+\theta)\wedge\mathbf{b} = (k+\theta)\wedge(\omega\mathbf{e}) = -\omega^2\mathbf{b}\,.$$

Therefore for $(k+\theta) \neq 0$ there are two roots $\omega = \pm|k+\theta|$. Each has a two dimensional eigenspace generated by taking $\mathbf{e} \perp (k+\theta)$ and $\mathbf{b} = \mp(k+\theta)\wedge\mathbf{e}$.

Return next to the general case.

Proposition 6.7. *There is an infinite dimensional space of θ-periodic solutions of* (6.5) *with eigenvalue $\lambda = 0$ consisting of E and B with vanishing* curls. *The $L^2_{\epsilon_0,\mu_0}(\mathbb{T}^3_\theta)$-orthogonal complement to this kernel satisfy* $\mathrm{div}(\epsilon_0 E) = \mathrm{div}(\mu_0 B) = 0$. *There is a constant C independent of θ so that θ-periodic E, B satisfy*

$$\begin{aligned}\|E,B\|_{H^1(\mathbb{T}^3_\theta)} \ \le\ C\Big(&\|\mathrm{curl}\ E\|_{L^2(\mathbb{T}^3_\theta)} \ +\ \|\mathrm{curl}\ B\|_{L^2(\mathbb{T}^3_\theta)} \\ &+\ \|\mathrm{div}(\epsilon_0(x)E)\|_{L^2(\mathbb{T}^3_\theta)} \ +\ \|\mathrm{div}\,(\mu_0(x)B)\|_{L^2(\mathbb{T}^3_\theta)} \ +\ \|E,B\|_{L^2(\mathbb{T}^3_\theta)}\Big)\,.\end{aligned} \tag{6.7}$$

For each θ the $\lambda = i\omega \neq 0$ for which (6.5) *has a nontrivial solution is a discrete set in $\mathbb{R}\setminus\{0\}$. Each eigenspace is a finite dimensional space of smooth functions. The value* 0 *is not an accumulation point of nonzero eigenvalues. The ω are not bounded above and are not bounded below.*

Proof. The estimate is the key. Write $u = e^{i\theta x}\widetilde{u}$ with periodic $\widetilde{u}$. Expand the latter in a Fourier series. The proof of the Lemma 3.3 yields (6.7).

Proposition 6.5 implies that the stationary solutions are curl free and their orthogonal complement is invariant under the flow by Maxwell's equations. Also that the complement consists of solutions satisfying $\mathrm{div}(\epsilon_0(x)E) = \mathrm{div}(\mu_0(x)B) = 0$.

Decompose $L^2_{\epsilon_0,\mu_0}(\mathbb{T}^3_\theta)$ as a Maxwell invariant direct sum of stationary and dynamic states

$$L^2_{\epsilon_0,\mu_0}(\mathbb{T}^3_\theta) \ =\ \mathrm{Ker}\,G\ \oplus_\perp\ \mathcal{H}_{\rm dyn}\,, \qquad \mathcal{H}_{\rm dyn}\ :=\ \big\{E,B\,:\,\mathrm{div}(\epsilon_0(x)E)\ =\ \mathrm{div}\mu_0(x)B\ =\ 0\big\}.$$

The $L^2_{\epsilon_0,\mu_0}(\mathbb{T}^3_\theta)$ unitary group e^{tG} restricts to a unitary group on $\mathcal{H}_{\rm dyn}$ whose anti selfadjoint generator is the restriction $G|_{\mathcal{H}_{\rm dyn}}$.

Estimate (6.7) implies that $(I + G|_{\mathcal{H}_{\rm dyn}})^{-1}$ is compact, hence has pure point spectrum tending to zero and total multiplicity in $\{|z| \ge \delta > 0\}$ finite for all $\delta > 0$. Therefore the spectrum of $G|_{\mathcal{H}_{\rm dyn}}$ is pure point and the total multiplicity in any bounded set is finite.

Commuting with derivatives yields an inductive proof of an H^s version of (6.7) for $1 \le s \in \mathbb{N}$,

$$\begin{aligned}\|E,B\|_{H^{s+1}(\mathbb{T}^3_\theta)} \ \le\ C(s)\Big(&\|\mathrm{curl}\ E\|_{H^s(\mathbb{T}^3_\theta)} \ +\ \|\mathrm{curl}\ B\|_{H^s(\mathbb{T}^3_\theta)} \\ &+\ \|\mathrm{div}(\epsilon_0(x)E)\|_{H^s(\mathbb{T}^3_\theta)} \ +\ \|\mathrm{div}\,(\mu_0(x)B)\|_{H^s(\mathbb{T}^3_\theta)} \ +\ \|E,B\|_{H^s(\mathbb{T}^3_\theta)}\Big)\,.\end{aligned} \tag{6.8}$$

The smoothness of eigenfunctions for eigenvalues $\lambda \neq 0$ follows.

It remains to show that the spectrum is unbounded above and unbounded below. Define P to be the bounded strictly positive selfadjoint multiplication operator $P(E,B) := (\epsilon\,E, \mu\,B)$. The eigenvalue equation is $Gu = i\omega P u$. It holds if and only if $v = P^{1/2}u$ satisfies $P^{-1/2}(G/i)P^{-1/2}v = \omega v$. Need to show that the eigenvalues are not bounded below and not bounded above. The ω are bounded below (resp. above) by $C \in \mathbb{R}$ if and

only if for all v so that $P^{-1/2}v$ belongs to the domain of G,

$$\langle P^{-1/2}(G/i)P^{-1/2}v\,,\,v\rangle\ \geq\ C\big\langle v\,,\,v\big\rangle\,,\qquad(\text{resp. }\leq)\,.$$

This holds if and only if for all u belonging to the domain of G,

$$\langle (G/i)u\,,\,u\rangle\ \geq\ C\big\langle P^{1/2}u\,,\,P^{1/2}u\big\rangle\,,\qquad(\text{resp. }\leq)\,.$$

This holds if and only if the spectral problem with $\epsilon_0=\mu_0=I$ has eigenvalues bounded below (resp. above). Example (6.6) shows by explicit computation that for $\epsilon_0=\mu_0=I$ the spectrum is unbounded in both directions. Thus the ω are not bounded below and not bounded above. □

Proposition 6.7 implies that the discrete spectrum at fixed θ consists of $\{0\}$ with infinite multiplicity and a discrete set of possibly multiple eigenvalues $\lambda=i\omega$ that we label according to their distance from the origin as in (1.5). Away from eigenvalue crossings, the functions $\theta\mapsto\omega_j(\theta)$ are real analytic. Rellich's theorem shows that away from the crossings the associated spectral projections $\Pi_j(\theta)$ are also real analytic in the sense that the unitary map of Proposition 6.4 intertwines them with an analytic family acting on $L^2_{\epsilon,\mu}(\mathbb{T}^3)$ (see [20]).

For θ fixed and an eigenvalue $\omega(\theta)\neq 0$ there is a finite dimensional space of eigenfunctions $e^{i\theta.x}(E_\theta(x),B_\theta(x))\in L^2(\mathbb{T}^3_\theta)$ and corresponding **Bloch plane wave** solutions of (1.1)

$$e^{i\omega(\theta)t}\ e^{i\theta.x}\ (E_\theta(x),B_\theta(x))\,.$$

We assume the constant multiplicity Hypothesis 1.4.

From the analytic dependence of the operators it follows that the $L^2(\mathbb{T}^3_\theta)$ orthogonal projection, $\Pi(\theta)\in\mathrm{Hom}\,(L^2(\mathbb{T}^3_\theta))$ onto the nullspace of $i\omega(\theta)A_0^0(y)-\sum_j A_j\partial_j$ is analytic and in particular of constant rank.

7. The purely periodic case

Fix $\underline{\theta}$ and a locally constant multiplicity eigenvalue $\omega(\theta)\neq 0$. Denote by $\mathbb{K}(\theta)$ the kernel of $\mathbb{L}(\omega(\theta),\theta,y,\partial_y)$ from Definitions 1.6. If $\theta\mapsto e^{i\theta.x}\psi(x,\theta)$ is a smooth function of θ on a neighborhood of $\underline{\theta}$ with values in $\mathbb{K}(\theta)$ then ψ is periodic with period 2π in x and smooth in its dependence on x,θ for $\theta\approx\underline{\theta}$. The function

$$e^{i\omega(\theta)t}\ e^{i\theta.x}\ \psi(x,\theta)$$

is a θ-periodic solution of Maxwell's equation in the periodic medium $\epsilon(x),\mu(x)$.

Scaling the periodic structure to $\epsilon(x/h),\mu(x/h)$ yields the corresponding rapidly oscillatory Bloch plane waves

$$e^{i\omega(\theta)t/h}\ e^{i\theta.x/h}\ \psi(x/h\,,\,\theta)\,.$$

For $a \in C_0^\infty(\mathbb{R}^3)$ and $h << 1$ the function $a((\theta - \underline{\theta})/h)$ is supported in the domain of definition of $\omega(\theta)$. Superposing nearby waves yields a Bloch wave packet for $h << 1$

$$\int_{[0,1[^3} a\Big(\frac{\theta-\underline{\theta}}{h}\Big)\ e^{i\omega(\theta)t/h}\ e^{i\theta.x/h}\ \psi(x/h\,,\,\theta)\ d\theta, \qquad a \in C_0^\infty(\mathbb{R}^3)\,.$$

Letting $\zeta := (\theta - \underline{\theta})/h$ yields the exact solutions

$$u^h \;=\; e^{i\underline{\theta}.x/h} \int_{\mathbb{R}^3} \psi\Big(\frac{x}{h}\,,\,\underline{\theta}+h\zeta\Big)\ e^{it\omega(\underline{\theta}+h\zeta)/h}\ e^{ix.\zeta}\ a(\zeta)\ d\zeta\,. \tag{7.1}$$

7.1. **The geometric optics time scale $\mathbf{t \sim 1}$.**

Definition 7.1. *Complementing Definition 1.5 the corresponding transport operator is defined by*

$$\mathbb{D}(\partial_t, \partial_x) \;:=\; \partial_t \;+\; \mathcal{V}.\partial_x\,.$$

The symbol is $\mathbb{D}(\tau, k) = \tau + \mathcal{V}.k$.

The Taylor series in h of infinite and finite orders respectively are,

$$\psi(y, \underline{\theta}+h\zeta) \;\sim\; \psi(y,\underline{\theta}) + \sum_{j\geq 1} h^j\ g_j(y,\zeta)\,, \quad \omega(\underline{\theta}+h\zeta) \;=\; \omega(\underline{\theta}) - \mathcal{V}h\zeta + h^2 k(h,\zeta)\,, \tag{7.2}$$

where the sum on the right hand side of $\sim$ is an asymptotic expansion as $h \to 0$, not a convergent series. Then,

$$\begin{aligned} e^{it\omega(\underline{\theta}+\epsilon\zeta)/h} \;&=\; e^{it\omega(\underline{\theta})/h}\ e^{-it\mathcal{V}.\zeta}\ e^{i(ht)k(h,\zeta)} \\ &=\; e^{it\omega(\underline{\theta})/h}\ e^{-it\mathcal{V}.\zeta}\ \Big(1 + \sum_{j\geq 1} (ht)^j k_j(h,\zeta)\Big)\,, \end{aligned} \tag{7.3}$$

where the last line uses a Taylor expansion of $s \mapsto e^{isk(h,\zeta)}$ about $s = 0$. Define

$$v(x) \;:=\; \int e^{ix.\zeta}\ a(\zeta)\ d\zeta\,.$$

Use (7.2) and (7.3) in (7.1) to find

$$u^h \;\sim\; e^{iS/h}\ w(h,t,x,x/h)\,, \qquad S \;:=\; \omega(\underline{\theta})t + x.\underline{\theta}\,, \tag{7.4}$$

and

$$w(h,t,x,y) \;\sim\; w_0(t,x,y) + hw_1(t,x,y) + \cdots$$

in the sense of Taylor series about $h = 0$. The leading term is

$$w_0(t,x,y) \;=\; v(x - \mathcal{V}t)\,\psi(y,\underline{\theta})\,.$$

The velocity is $\mathcal{V} = -\nabla_\theta \omega(\underline{\theta})$. It is not at all obvious from this definition that $\mathcal{V}$ does not exceed the speed of light. As our media are anisotropic, the speed may depend on direction. We recall the algorithm giving such anisotropic speeds (see [23], [19], [25]).

Denote by (τ,ξ) the dual variable of (t,y). The characteristic polynomial of Maxwell's equations is

$$p(y,\tau,\xi)\ :=\ \det\left(\tau\begin{pmatrix}\epsilon_0(y) & 0\\ 0 & \mu_0(y)\end{pmatrix}+\begin{pmatrix}0 & -\xi\wedge\\ \xi\wedge & 0\end{pmatrix}\right).$$

Its roots $\tau(y,\xi)$ for ξ real define the characteristic variety. Define extreme roots $\tau_{\max}(y,\xi)$ by

$$\tau_{\max}(y,\xi)\ :=\ \max\big\{\tau\ :\ p(y,\tau,\xi)=0\big\}.$$

with a similar defintion for $\tau_{\min}(y,\xi)$. The function $\tau_{\max}$ is positive homogeneous of degree one in ξ, and,

$$\tau_{\max}(y,-\xi)\ =\ -\,\tau_{\min}(y,\xi)\,. \tag{7.5}$$

The set of velocities that do not exceed the speed of propagation is a convex set of vectors $\mathbf{v}$ whose support function equal $\tau_{\max}(y,-\xi)$, equivalently

$$\cap_{\xi\in\mathbb{R}^3\setminus\{0\}}\ \big\{\mathbf{v}\ :\ \xi.\mathbf{v}\ \le\ \tau_{\max}(y,-\xi)\big\}\,.$$

In case of constant and isotropic permittivities $\tau_{\max}(y,-\xi)\ =\ |\xi|/\sqrt{\epsilon\mu}$ yielding the classic formula for the speed of light.

Define

$$\tau_{\max}(\xi)\ :=\ \max_{y\in\mathbb{T}^3}\ \tau_{\max}(y,\xi)\,.$$

Then $\tau_{\max}(-\xi)$ is the largest speed limit for $\xi.\mathbf{v}$.

Theorem 7.2. *The group velocity $\mathcal{V}$ from Definition 1.5 respects the maximum speed of propagation for each $\xi\neq0$, precisely for all $0\neq\xi\in\mathbb{R}^3$, $\xi.\mathcal{V}\le\tau_{\max}(-\xi)$.*

Proof. The Bloch spectral eigenvalue problem for periodic as opposed to θ-periodic functions is

$$i\,\omega(\theta)\begin{pmatrix}\epsilon_0(y) & 0\\ 0 & \mu_0(y)\end{pmatrix}\Pi(\theta)+\begin{pmatrix}0 & -(i\theta+\partial_y)\wedge\\ (i\theta+\partial_y)\wedge & 0\end{pmatrix}\Pi(\theta)=0 \tag{7.6}$$

where $\Pi(\theta)$ is the projector (of constant rank) on the eigensubspace of the eigenvalue $\omega(\theta)$. The constant multiplicity hypothesis implies that the eigenvalue and the spectral projector are analytic in a vicinity of $\underline{\theta}$. Differentiating (7.6) with respect to θ in the direction of a covector ξ and multiplying on the left by $\Pi(\theta)$ yields

$$\xi.\nabla_\theta\omega(\theta)\ \Pi(\theta)\begin{pmatrix}\epsilon_0(y) & 0\\ 0 & \mu_0(y)\end{pmatrix}\Pi(\theta)+\Pi(\theta)\begin{pmatrix}0 & -\xi\wedge\\ \xi\wedge & 0\end{pmatrix}\Pi(\theta)=0\,. \tag{7.7}$$

Take any eigenfunction $E(y),B(y)$, normalized by

$$\int_{\mathbb{T}^3}\langle\begin{pmatrix}\epsilon_0(y) & 0\\ 0 & \mu_0(y)\end{pmatrix}\begin{pmatrix}E(y)\\ B(y)\end{pmatrix},\begin{pmatrix}E(y)\\ B(y)\end{pmatrix}\rangle\ dy=1\,. \tag{7.8}$$

The quadratic form associated to (7.7) yields

$$\xi.\nabla_\theta \omega(\theta) \;+\; \int_{\mathbb{T}^3} \langle \begin{pmatrix} 0 & -\xi\wedge \\ \xi\wedge & 0 \end{pmatrix} \begin{pmatrix} E(y) \\ B(y) \end{pmatrix}, \begin{pmatrix} E(y) \\ B(y) \end{pmatrix} \rangle \, dy \;=\; 0\,. \tag{7.9}$$

The $-\tau(y,\xi)$ are the eigenvalues with respect to the postitive definite matrix diag $\{\epsilon_0(y)\,,\,\mu_0(y)\}$ so the min-max characterization implies that for each y

$$\begin{aligned} \langle \begin{pmatrix} 0 & -\xi\wedge \\ \xi\wedge & 0 \end{pmatrix} \begin{pmatrix} E(y) \\ B(y) \end{pmatrix}, \begin{pmatrix} E(y) \\ B(y) \end{pmatrix} \rangle \;&\leq\; -\,\tau_{\min}(y,\xi)\, \langle \begin{pmatrix} \epsilon_0(y) & 0 \\ 0 & \mu_0(y) \end{pmatrix} \begin{pmatrix} E(y) \\ B(y) \end{pmatrix}, \begin{pmatrix} E(y) \\ B(y) \end{pmatrix} \rangle \\ &\leq\; -\,\tau_{\min}(\xi)\, \langle \begin{pmatrix} \epsilon_0(y) & 0 \\ 0 & \mu_0(y) \end{pmatrix} \begin{pmatrix} E(y) \\ B(y) \end{pmatrix}, \begin{pmatrix} E(y) \\ B(y) \end{pmatrix} \rangle\,. \end{aligned}$$

Integrating and using (7.8) and (7.9) proves

$$\xi.\nabla_\theta \omega(\theta) \;-\; \tau_{\min}(\xi) \;\geq\; 0\,,$$

which is the desired relation since $\mathcal{V} = -\nabla_\theta \omega(\theta)$. □

An analogous result to Theorem 7.2 is proved in [3], where a bound on the group velocity for scalar wave equations is given.

7.2. **The diffractive time scale t $\sim$ 1/h.** In (7.3), the expansion parameter is ht so when ht is not small, the approximation is not appropriate. For the diffractive scale $ht \sim 1$, one needs a refinement. Take the next term in the Taylor expansion in the exponent. Denote by q the symmetric quadratic expression

$$q(\zeta,\zeta) \;:=\; \sum_{i,j=1}^{3} \frac{\partial^2 \omega(\underline{\theta})}{\partial\theta_i \partial\theta_j}\, \zeta_i\, \zeta_j\,. \tag{7.10}$$

Then,

$$\omega(\underline{\theta} + \epsilon\zeta) \;=\; \omega(\underline{\theta}) \;-\; h\mathcal{V}.\zeta \;+\; h^2 q(\zeta,\zeta)/2 \;+\; h^3 \sum_{j\geq 0} h^j \beta_j(\zeta)\,,$$

and,

$$e^{i\omega(\underline{\theta}+h\zeta)t/h} \;=\; e^{i\omega(\underline{\theta})t/h}\; e^{-it\mathcal{V}.\zeta}\; e^{ihtq(\zeta,\zeta)/2}\; e^{ih(ht)\sum_{j\geq 0} h^j \beta_j(\zeta)}\,.$$

Introduce the slow time $\mathcal{T} = ht$. The exact solution has the form

$$e^{2\pi i S/h}\, \widetilde{W}(h, ht, x - \mathcal{V}t, x/h)\,, \qquad S = \omega(\underline{\theta})t + \underline{\theta}.x\,,$$

$$\widetilde{W}(h, \mathcal{T}, x, y) \;:=\; \int \psi(y, \underline{\theta} + h\zeta)\; e^{i\mathcal{T}q(\zeta,\zeta)/2}\; e^{ih\mathcal{T}\sum_{j\geq 0} h^j\beta_j(\zeta)}\; e^{ix.\zeta}\; a(\zeta)\, d\zeta\,.$$

Taylor expansion in h yields

$$e^{ih\mathcal{T}\sum_{j\geq 0} h^j\beta_j(\zeta)} \;=\; \Big(1 + \sum_{j\geq 1} h^j\, r_j(\mathcal{T},\zeta)\Big)\,. \tag{7.11}$$

Using (7.2) and (7.11) in the definition of $\widetilde{W}$ shows that

$$\widetilde{W}(h, \mathcal{T}, x, y) \ \sim \ \sum_{j \geq 0} h^j \, \widetilde{w}_j(\mathcal{T}, x, y)\,, \tag{7.12}$$

with

$$\widetilde{w}_0(\mathcal{T}, x, y) \ = \ \psi(y, \underline{\theta}) \int e^{i\mathcal{T}\, q(\zeta,\zeta)/2} \, e^{ix.\zeta} \, a(\zeta)\, d\zeta\,. \tag{7.13}$$

This shows that the solution has an asymptotic expansion of the form

$$e^{iS/h} \, \widetilde{W}(h, ht, x - \mathcal{V}t, x/h)\,,$$

with $\widetilde{W}$ satisfying (7.12).

Equation (7.13) implies that $\widetilde{w}_0$ is a tempered solutions (with values in $\mathbb{K}$) of the Schrödinger equation

$$i\, \partial_{\mathcal{T}} \widetilde{w}_0 \ - \ \frac{1}{2}\, \partial^2_\theta \omega(\underline{\theta})\big(\partial_x, \partial_x\big)\, \widetilde{w}_0 \ = \ 0\,. \tag{7.14}$$

Though the function $\widetilde{w}_0$ takes values in the finite dimensional space $\mathbb{K}$ the equation (7.14) is scalar. The constant rank hypothesis is crucial here.

8. Bloch wave packets on a modulated background and $t = \mathcal{O}(1)$

This section considers solutions of the Maxwell's equations for times $t = \mathcal{O}(1)$. This time scale is an essential first step in treating the diffractive case. In order that the asymptotic description be nontrivial we allow lower order terms in the equations. In particular this includes the case of a possibly conducting medium with Ohm's law dissipation,

$$\epsilon(x, x/h)\, \partial_t E \ = \ \mathrm{curl}\, B \ - \ \sigma(t, x, x/h) E, \qquad \mu(x, x/h)\, \partial_t B \ = \ -\mathrm{curl}\, E\,. \tag{8.1}$$

Here $\sigma(t, x, y)$ is a nonnegative symmetric matrix valued function. The physics modelled is that where $\sigma \neq 0$ the medium reacts instantaneously to the field E by generating a current $J = \sigma\, E$. Such an assumption is realistic only when the field E varies little on the time scales associated to the motion of electrons. The associated energy dissipation law is

$$\partial_t \int_{\mathbb{R}^3} \langle \epsilon E\,,\, E\rangle \ + \ \langle \mu B\,,\, B\rangle \; dx \ = \ - \int_{\mathbb{R}^3} \big\langle \sigma E\,,\, E \big\rangle \; dx \ \leq \ 0\,.$$

The lower order tem is $\mathcal{O}(1)$ and appreciably affects the fields for times $t = \mathcal{O}(1)$. In this section only, we replace Hypothesis 1.1 by the following that allows (8.1). The perturbations are larger by a factor h^{-1} than in the diffractive case.

Hypothesis 8.1. $T = \mathcal{O}(1)$ **hypothesis.**

$$0 \ = \ P^h(t, x, \partial_t, \partial_x) u^h \ := \ \partial_t(A_0^h u^h) \ + \ \sum_{j=1}^{3} A_j \partial_{x_j} u^h \ + \ M^h\, u^h\,. \tag{8.2}$$

The coefficients A_j, for $j = 1, 2, 3$, are the constant 6×6 matrices

$$A_1 := \begin{pmatrix} 0 & J_1 \\ -J_1 & 0 \end{pmatrix}, \quad A_2 := \begin{pmatrix} 0 & J_2 \\ -J_2 & 0 \end{pmatrix}, \quad A_3 := \begin{pmatrix} 0 & J_3 \\ -J_3 & 0 \end{pmatrix}, \tag{8.3}$$

$$J_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J_2 := \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J_3 := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{8.4}$$

The coefficient A_0^h and M^h are of the form

$$A_0^h(t,x) \;=\; A_0^0(x/h) + hA_0^1(t,x,x/h)\,, \quad M^h = M(t,x,x/h)\,, \tag{8.5}$$

where A_0^0 and A_0^1 satisfy Hypothesis 1.1.

Remark 8.2. *It follows that the growth rate $C(m,h)$ from Theorem 4.1 (and Remark 4.3) is of order $\mathcal{O}(1)$ as $h \to 0$.*

Motivated by the special case of purely periodic media in Section 7.4 and the linear case of Lax [22] (see also [25]) try the *ansatz* of two scale WKB type,

$$v^h(t,x) \;:=\; e^{iS(t,x)/h} W\Big(h,t,x,\frac{x}{h}\Big), \qquad W\big(h,t,x,y\big) \;=\; w_0(t,x,y) + hw_1(t,x,y)\,, \tag{8.6}$$

where the $w_j(t,x,y)$ are periodic functions of y with period 2π. The case when S is a linear function of (t,x) is our principal interest

$$S(t,x) \;=\; \omega\, t + \theta\,.x\,, \qquad (\omega,\theta) \in \mathbb{R}^{1+3} \setminus \{0\}\,.$$

For this phase the rays will be parallel straight lines and one finds Schrödinger type equations at the diffractive scale $t = \mathcal{O}(1/h)$.

Three identities are at the heart of checking the accuracy of the *ansatz*

$$\begin{aligned}
\partial_t\Big[A_0^h\, e^{iS(t,x)/h}\, W\big(h,t,x,y\big)\Big] \;&=\; e^{iS(t,x)/h}\Big(\frac{i\omega}{h} + \partial_t\Big)\Big[A_0^h\, W\big(h,t,x,y\big)\Big]\,,\\
\mathrm{curl}_{\,x}\Big[e^{iS(t,x)/h}\, W\big(h,t,x,y\big)\Big] \;&=\; e^{iS(t,x)/h}\,\Big(\,\partial_x + \frac{i\theta}{h}\,\Big) \wedge W\big(h,t,x,y\big)\,,\\
\frac{1}{h}\mathrm{curl}_{\,y}\Big[e^{iS(t,x)/h}\, W\big(h,t,x,y\big)\Big] \;&=\; e^{iS(t,x)/h}\,\frac{1}{h}\,\partial_y \wedge W\big(h,t,x,y\big)\,.
\end{aligned} \tag{8.7}$$

These yield

$$P^h(t,x,\partial_t,\partial_x)v^h \;=\; e^{iS(t,x)/h}\, Z^h(t,x,x/h)\,, \qquad \text{with} \tag{8.8}$$

$$Z^h := \left[\Big(\frac{i\omega}{h} + \partial_t\Big)A_0^h - \begin{pmatrix} 0 & \big(\,\partial_x + \frac{i\theta}{h}\,\big)\wedge \\ -\big(\,\partial_x + \frac{i\theta}{h}\,\big)\wedge & 0 \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{h}\,\partial_y\wedge \\ -\frac{1}{h}\,\partial_y\wedge & 0 \end{pmatrix} + M\right] W.$$

Then

$$Z^h(t,x,y) \;=\; h^{-1}\, r_{-1} \;+\; r_0 \;+\; h\, r_1 \;+\; h^2 r_2\,, \qquad r_j \;=\; r_j(t,x,y)\,. \tag{8.9}$$

Since one substitutes $y = x/h$, it would suffice to satisfy $r_j = 0$ on the subspace of (t, x, y) with x parallel to y. We achieve the more ambitious goal of choosing the w_j so that $r_{-1} = r_0 = 0$ everywhere.

8.1. **The leading order term.** The leading two orders in Z^h are

$$h^{-1} r_{-1} \ + \ r_0 \ = \ h^{-1}\, \mathbb{L}(\omega, \theta, y, \partial_y) W \ + \ \Big(\mathbb{M}(\omega, \theta, y, \partial_t, \partial_x, \partial_y) \ + \ i\omega A_0^1 + \ M \Big) W\,,$$

where $\mathbb{L}$ is from (1.6) and

$$\mathbb{M}(y, \partial_t, \partial_x) \ := \ A_0^0(y)\partial_t - \begin{pmatrix} 0 & \partial_x \wedge \\ -\partial_x \wedge & 0 \end{pmatrix}. \tag{8.10}$$

The highest order term in (8.9) is

$$r_{-1} \ = \ \mathbb{L}(\omega, \theta, y, \partial_y)\, w_0\,. \tag{8.11}$$

In order that $r_{-1} = 0$ have nontrivial solutions, it is necessary that

$$\ker \mathbb{L}(\omega, \theta, y, \partial_y) \ \neq \ \{0\}\,.$$

According to the Floquet-Bloch theory of Section 6, $\mathbb{L}(\omega, \theta, y, \partial_y)$ has a nontrivial kernel on periodic functions if and only $i\omega = i\omega(\theta)$ is an eigenvalue of (1.4). From now on we make the choice of $\underline{\theta}$ and $\omega(\underline{\theta})$ so that the constant multiplicity hypothesis 1.4 is satisfied.

Definition 8.3. *In addition to Definition 1.6 denote by* $\mathbb{Q} \in \mathrm{Hom}\big(H^s(\mathbb{T}^3_y); H^{s+1}(\mathbb{T}^3_y)\big)$ *the partial inverse of* $\mathbb{L}$ *defined by*

$$\mathbb{Q}\,\Pi = \Pi\,\mathbb{Q} = 0\,, \qquad \mathbb{Q}\mathbb{L} \ = \ \mathbb{L}\mathbb{Q} \ = \ I - \Pi\,.$$

The equation $r_{-1} = 0$ is equivalent to $w_0 \in \mathbb{K} := \ker \mathbb{L}$, that is

$$\Pi\, w_0 = w_0\,. \tag{8.12}$$

Using the definitions of $\mathbb{L}$ and $\mathbb{M}$, the term r_0 is given by,

$$r_0 \ = \ \mathbb{L} w_1 \ + \ \big(\mathbb{M} + \ i\omega A_0^1 + M\big) w_0, \tag{8.13}$$

so $r_0 = 0$ if and only if,

$$\mathbb{L}\, w_1 \ + \big(\mathbb{M} + \ i\omega A_0^1 + M\big) w_0 \ = \ 0\,. \tag{8.14}$$

Equation (8.14) involves both w_0 and w_1. Since $\mathbb{L}$ is selfadjoint on $L^2(\mathbb{T}^3)$ its range is perpendicular to its kernel so $\Pi\,\mathbb{L} = 0$. This is true only because Π is the orthogonal projection on the kernel $\mathbb{K}$ for the $L^2(\mathbb{T}^3)$ scalar product (not for the other scalar product in Definition 1.6). The equation $\Pi\, r_0 = 0$ yields an equation for w_0 alone,

$$\Pi\,\big(\mathbb{M} \ + \ i\omega A_0^1 + \ M\big) w_0 \ = \ 0\,.$$

Taking into account (8.12), this is equivalent to

$$\Pi\big(\mathbb{M} \ + \ i\omega A_0^1 + \ M\big)\Pi\, w_0 \ = \ 0\,. \tag{8.15}$$

Proposition 8.4. *For any $w(t,x,y) \in C^\infty$,*

$$\Pi\,\mathbb{M}\,\Pi\,w \;=\; \Pi\,A_0^0\,\Pi\,\big(\partial_t + \mathcal{V}.\partial_x\big)\,w\,, \tag{8.16}$$

with the group velocity $\mathcal{V}$ from Definition 1.5. The operator $\Pi\,A_0^0\,\Pi$ is a linear isomorphism of $\mathbb{K}$ to itself.

Proof. Prove the last sentence first. Since $\mathbb{K}$ is finite dimensional, it suffices to prove injectivity. If $\Pi\,A_0\,\Pi\,k = 0$ then

$$0 \;=\; \big\langle \Pi\,A_0\,\Pi\,k\,,\,k\big\rangle \;=\; \big\langle A_0\,\Pi\,k\,,\,\Pi k\big\rangle \;\geq\; c\,\|\Pi k\|^2\,, \qquad c>0\,,$$

since A_0 is strictly postive. Therefore $k = \Pi k = 0$ proving injectivity.

From the definition of Π and $\mathbb{M}$ one automatically has for arbitrary w,

$$\Pi\,\mathbb{M}\,\Pi\,w \;=\; \Big(a_0\,\partial_t \;+\; \sum_{j=1}^3 a_j\,\frac{\partial}{\partial x_j}\Big)\Pi\,w\,,$$

with matrices $a_j(y)$. It suffices to compute the a_j. This is done by computing the differential operator on the test functions $t\,\psi$, and $x_j\psi$, with $\psi \in \mathbb{K}$. Applying to $t\psi$ and setting $t=0$ yields

$$a_0\,\Pi \;=\; \Pi\,A_0^0\,\Pi\,. \tag{8.17}$$

Applying to $x_j\psi$ and setting $x_j = 0$ yields

$$a_j\,\Pi \;=\; -\,\Pi\begin{pmatrix} 0 & e_j\wedge \\ -e_j\wedge & 0\end{pmatrix}\Pi\,, \qquad \{e_1,e_2,e_3\} \text{ is the standard basis of } \mathbb{R}^3\,. \tag{8.18}$$

The identification of a_j from (8.18) and (8.17) requires first order perturbation theory as in (8.19) of the next proposition. Second order perturbation theory as in (8.20) is needed for diffractive geometric optics. The identites are proved by differentiating the identities $\Pi\,L = 0$ and $L\,\Pi = 0$. We refer the reader to [2], [25] for detailed proof.

Proposition 8.5. *Suppose that $\underline{\theta}$ and ω satisfy the constant multiplicity hypothesis 1.4. Suppose that the coefficient A_0^0 and θ depend smoothly on a parameter α with their unperturbed values attained at $\underline{\alpha}$. With $'$ denoting differentiation with respect to α, the following perturbation formulas hold,*

$$\Pi\,\mathbb{L}'\,\Pi \;=\; 0\,, \tag{8.19}$$

and

$$\Pi\,\mathbb{L}''\,\Pi \;-\; 2\,\Pi\,\mathbb{L}'\,Q\,\mathbb{L}'\,\Pi \;=\; 0\,. \tag{8.20}$$

Returning to the formula for a_j, use (8.19) with α equal to the j^{th} component of θ and $' = \partial/\partial\theta_j$. Then,

$$\mathbb{L}' \;=\; i\,\frac{\partial\omega}{\partial\theta_j}\,A_0^0(y) \;-\; \begin{pmatrix} 0 & i\,e_j\wedge \\ -i\,e_j\wedge & 0\end{pmatrix}.$$

The above identity in combination with (8.19) and (8.18) yeilds

$$a_j\,\Pi \;=\; -\,\Pi\,A_0^0(y)\,\Pi\,\frac{\partial\omega}{\partial\theta_j}\,.$$

This together with (8.17) completes the proof of Proposition 8.4. ☐

Recall Definition 1.7. The map γ inherits the regularity of the coefficients,

$$\partial_{t,x}^{\alpha}\gamma \in L^\infty\big(\mathbb{R}^{1+3}\,;\,\mathrm{Hom}\,\mathbb{K}\big)\,. \tag{8.21}$$

Then $r_{-1} = \Pi r_0 = 0$ exactly when $w_0 = \Pi w_0$ satisfies the transport equation

$$\Big(\partial_t \;+\; \mathcal{V}.\partial_x \;+\; \gamma(t,x)\Big)w_0 \;=\; 0\,. \tag{8.22}$$

The equation $r_0 = 0$ is equivalent to the pair

$$\Pi\,r_0 \;=\; 0\,,\qquad \mathbb{Q}\,r_0 \;=\; 0\,.$$

Equation (8.13) shows that $\mathbb{Q}\,r_0 = 0$ if and only if

$$\big(I-\Pi\big)w_1 \;=\; -\mathbb{Q}\big(\mathbb{M} +\; i\omega A_0^1 + M\big)w_0\,. \tag{8.23}$$

The choice of $\Pi\,w_1$ does not influence r_1, r_0. Choose

$$\Pi\,w_1 = 0\,. \tag{8.24}$$

Theorem 8.6. *If $g \in C^\infty(\mathbb{R}^3;\mathbb{K})$ there is a unique $w_0 \in C^\infty(\mathbb{R}; H^\infty(\mathbb{R}^3;\mathbb{K}))$ satisfying (8.22) with $w_0(0) = g$. Define w_1 and v^h by (8.23), (8.24) and (8.6) respectively. If u^h is the exact solution of $P^h u^h = 0$ with $u^h\big|_{t=0} = v^h\big|_{t=0}$, then for all α*

$$\sup_{t\in[0,T]}\;\big\|(h\,\partial_{t,x})^\alpha(u^h - v^h)\big\|_{L^2(\mathbb{R}^3)} \;\le\; C(\alpha)\,h\,,\qquad 0<h<1\,.$$

Proof. **I. Estimate for $P^h v^h$.** Use (8.8) together with (8.9) and the fact that the equations satisified by the functions w_j guarantee that $r_{-1} = r_0 = 0$. Therefore for $h \in]0,1[$

$$\big\|\partial_{t,x,y}^{\alpha} Z^h\big\|_{L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{T}^3)} \;\le\; C(\alpha)\,h,\qquad \mathrm{supp}\,Z^h \subset \big\{(t, x+\mathcal{V}t, y)\;:\; x\in\;\mathrm{supp}\;g\big\}\,.$$

This implies the fundamental residual estimate

$$\big\|(h\,\partial_{t,x})^\alpha\big(P^h v^h\big)\big\|_{L^\infty([0,T]\times\mathbb{R}^3)} \;\le\; C(\alpha)\,h,\qquad h\in]0,1[\,.$$

Using the compact support one has

$$\big\|(h\,\partial_{t,x})^\alpha\big(P^h v^h\big)\big\|_{L^\infty([0,T];L^2(\mathbb{R}^3))} \;\le\; C(\alpha)\,h,\qquad h\in]0,1[\,. \tag{8.25}$$

II. Stability for P^h. For $T>0$ and m fixed Theorem 4.1 shows that there is a constant $C = C(T,m)$ so that for $0\le t\le T$ and $h\in]0,1]$

$$\big\|w(t)\big\|_{m,h} \;\le\; C\,\Big(\big\|w(0)\big\|_{m,h} \;+\; \int_0^t \big\|P^h\,w(s)\big\|_{m,h}\,ds\Big)\,. \tag{8.26}$$

III. Combining. Apply (8.26) to $w^h := u^h - v^h$ to find

$$\begin{aligned}\big\|(u^h - v^h)(t)\big\|_{m,h} \;&\le\; C\Big(\big\|(u^h - v^h)(0)\big\|_{m,h} \;+\; \int_0^t \big\|P^h\,(u^h - v^h)(s)\big\|_{m,h}\,ds\Big)\\ &=\; C\Big(\big\|(u^h - v^h)(0)\big\|_{m,h} \;+\; \int_0^t \big\|P^h\,v^h(s)\big\|_{m,h}\,ds\Big)\\ &=\; C\Big(\big\|(u^h - v^h)(0)\big\|_{m,h} \;+\; \mathcal{O}(h)\Big).\end{aligned}$$

where the last step uses the residual estimate (8.25).

It remains to show that $w^h = u^h - v^h$ satisfies for for all m,

$$\|w^h(0)\|_{m,h} \;=\; \mathcal{O}(h), \qquad h \to 0\,. \tag{8.27}$$

To do that it suffices to show that for all $0 \le k \in \mathbb{N}$ and s,

$$\|(h\partial_t)^k w^h(0)\|_{H^s_h(\mathbb{R}^3)} \;=\; \mathcal{O}(h), \qquad h \to 0\,. \tag{8.28}$$

Since the initial values vanish, the case $k = 0$ is automatic. Prove (8.28) by induction on k. Suppose (8.28) known for $0 \le k \le \underline{k}$ and all s. We prove it for $\underline{k}+1$ and all s. Begin with the identity

$$h(A_0^h)^{-1}P^h w^h \;=\; h(A_0^h)^{-1}\big(A_0^h\partial_t w^h + (\partial_t A_0^h)w^h + \sum_j A_j\partial_j w^h + M^h w^h\big).$$

The first term on the right is $h\partial_t w^h$. Therefore

$$(h\partial_t)^{\underline{k}+1}w^h = (h\partial_t)^k h\partial_t w^h = (h\partial_t)^{\underline{k}}(A_0^h)^{-1}\,h\,\Big(P^h - (\partial_t A_0^h) - \sum_j A_j\partial_j - M^h\Big)w^h.$$

The inductive hypothesis implies that $\|(h\,\partial_t)^{\underline{k}+1}u(0)\|_{H^s_h(\mathbb{R}^3)} = \mathcal{O}(h)$. This completes the inductive proof of (8.28) and therefore the proof of the theorem. □

9. Bloch wave packets on a modulated background and $t = \mathcal{O}(1/h)$

This section considers solutions of the Maxwell's equations with coefficients and lower order terms satisfying Hypothesis 1.1 so as to become pertinent at $t \sim 1/h$.

Remark 9.1. *With permittivities satisfying Hypothesis 1.1, the growth rate $C(m,h)$ from Theorem 4.1 satisfies $C(m,h) \le c(m)\,h$ with a constant $c(m)$ independent of $h \in]0,h[$. The time evolution is uniformly bounded so long as the product $t \times h$ remains bounded.*

Motivated by the special case of purely periodic media in Section 7.4, the *ansatz* expected to be valid for times $t = \mathcal{O}(1/h)$ is of two scale WKB type,

$$v^h(t,x) \;:=\; e^{iS(t,x)/h}W\Big(h, ht, t, x, \frac{x}{h}\Big), \qquad S(t,x) \;=\; \omega\,t + \theta\,.x\,, \tag{9.1}$$

$$W\big(h,\mathcal{T},t,x,y\big) \;=\; w_0(\mathcal{T},t,x,y) \;+\; hw_1(\mathcal{T},t,x,y) \;+\; h^2 w_2(\mathcal{T},t,x,y),$$

where the $w_j(\mathcal{T}, t, x, y)$ are periodic functions of y with period 2π. To go further in time requires the additional corrector w_2. In order to preserve the relative ordering of the size of the terms for $t = \mathcal{O}(1/h)$ it is crucial to insist that *the $w_j(\mathcal{T}, t, x, y)$ are sublinear in t,*

$$\lim_{t\to\infty} \frac{w_j(\mathcal{T}, t, x, y)}{t} \;=\; 0$$

uniformly in $\mathcal{T}, x, y$. We construct profiles satisfying a stronger hypothesis.

Compute using (8.7)

$$P^h(t,x,\partial_t,\partial_x)v^h \;=\; e^{iS(t,x)/h}\, Z^h(h\mathcal{T}, t, x, x/h)\,, \qquad \text{with} \tag{9.2}$$

$$\begin{aligned} Z^h(\mathcal{T},t,x,y) \;:=\; & \Bigg[\Big(\frac{i\omega}{h} + \partial_t + h\partial_{\mathcal{T}}\Big)A_0^h - \\ & \begin{pmatrix} 0 & \big(\partial_x + i\theta/h\big)\wedge \\ -\big(\partial_x + i\theta/h\big)\wedge & 0 \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{h}\,\partial_y\wedge \\ -\frac{1}{h}\,\partial_y\wedge & 0 \end{pmatrix} + hM\Bigg] W\,. \end{aligned} \tag{9.3}$$

Then

$$Z^h(t,x,y) \;=\; h^{-1}\, r_{-1} \,+\, r_0 \,+\, h\, r_1 \,+\, h^2\, r_2 \,+\, h^3\, r_3\,, \qquad r_j \;=\; r_j(t,x,y)\,. \tag{9.4}$$

As in the case of $t = \mathcal{O}(1)$, we achieve the ambitious goal of choosing the w_j so that the leading $r_{-1} = r_0 = r_1 = 0$ everywhere. This reduces the residual to $\mathcal{O}(h^2)$ which allows us to justify for times $t = \mathcal{O}(1/h)$.

One has

$$r_{-1} \;=\; \mathbb{L}w_0 \qquad \text{and} \qquad r_0 \;=\; \mathbb{L}w_1 + \mathbb{M}w_0$$

with $\mathbb{L}$ and $\mathbb{M}$ defined by (1.6) and (8.10) respectively. The next coefficient is

$$\begin{aligned} r_1 \;&= \partial_t A_0^0 w_1 + \partial_\tau(A_0^0 w_0) + i\omega(A_0^0 w_2 + A_0^1 w_0) - \begin{pmatrix} 0 & \partial_x\wedge \\ -\,\partial_x\wedge & 0 \end{pmatrix} w_1 \\ &\qquad\qquad - \begin{pmatrix} 0 & \big(i\theta + \partial_y\big)\wedge \\ -\,\big(i\theta + \partial_y\big)\wedge & 0 \end{pmatrix} w_2 + Mw_0 \\ &= \mathbb{L}w_2 + \mathbb{M}w_1 + \mathbb{N}w_0\,, \end{aligned}$$

where

$$\mathbb{N} \;:=\; \partial_{\mathcal{T}} A_0^0 + i\omega A_0^1 + M\,. \tag{9.5}$$

As in Section 9, $r_{-1} = 0$ and $\Pi\; r_0 = 0$ if and only if

$$\Pi w_0 = w_0\,, \qquad \text{and} \qquad (\partial_t + \mathcal{V}.\partial_x) w_0 = 0\,,$$

with $\mathcal{V}$ as in Definition 1.5. Thus there is a reduced $\mathbb{K}$ valued profile $\widetilde{w}_0$ such that

$$w_0(\mathcal{T}, t, x, y) \;=\; \widetilde{w}_0(\mathcal{T}, x - \mathcal{V}t, y)\,.$$

In order to determine $\widetilde{w}_0$ one needs a dynamic equation in $\mathcal{T}$. The equation $\mathbb{Q}\ r_0 = 0$ yields

$$(I - \Pi)w_1 \ = \ -\mathbb{Q}\ \mathbb{M}\ w_0\,, \tag{9.6}$$

and thus

$$(\partial_t + \mathcal{V}.\partial_x)\big(I - \Pi\big)w_1 \ = \ 0\,. \tag{9.7}$$

The equation $\Pi r_1 = 0$ yields the Schrödinger equation determining the dynamics of $\widetilde{w}_0$. Plugging (9.6) into the equation $\Pi r_1 = 0$ yields

$$\Pi\ \mathbb{M}\ \Pi w_1 - \Pi\ \mathbb{M}\ \mathbb{Q}\ \mathbb{M}\ \Pi w_0 + \Pi\ \mathbb{N}\ \Pi w_0 = 0\,. \tag{9.8}$$

Proposition 8.4 and (9.7) imply that equation (9.8) is equivalent to

$$\big(\Pi\ \mathbb{N}\ \Pi - \Pi\ \mathbb{M}\ \mathbb{Q}\ \mathbb{M}\ \Pi\big)w_0 \ = \ -\Pi\ A_0^0\ \Pi\big(\partial_t + \mathcal{V}.\partial_x\big)\Pi w_1\,. \tag{9.9}$$

The next proposition identifies the operator on the left hand side of (9.9).

Proposition 9.2. *On smooth functions $w(\mathcal{T}, t, x, y)$ that satisfy $(\partial_t + \mathcal{V}.\partial_x)w = 0$,*

$$\Big(\Pi\,\mathbb{N}\,\Pi \ - \ \Pi\,\mathbb{M}\,Q\,\mathbb{M}\,\Pi\Big)\,w \ = \ \Pi A_0^0 \Pi\Big[\partial_{\mathcal{T}} + \frac{1}{2} i\ \partial_\theta^2 \omega(\partial_x, \partial_x) + \gamma(t,x)\Big]\,w\,. \tag{9.10}$$

Remark 9.3. *It is surprising to find that the operator*

$$(\Pi A_0^0 \Pi)^{-1}\big(\Pi\,\mathbb{N}\,\Pi \ - \ \Pi\,\mathbb{M}\,Q\,\mathbb{M}\,\Pi\big)$$

acting on $\mathbb{K}$ valued functions has leading terms that are scalar. Coupling only occurs through the zero order term γ. A related zero order coupling occurs in §6 of [4]

Proof. With $k = (k_1, k_2, k_3) \in \mathbb{R}^3$ fixed, $\theta := \underline{\theta} + \alpha k$, $\alpha \in \mathbb{R}$, differentiate $\mathbb{L}(\omega, \theta, y, k)$ with respect to α to find

$$\begin{aligned}
&\mathbb{L}' = ikA_0^0 \partial_\theta \omega - \begin{pmatrix} 0 & ik\wedge \\ -ik\wedge & 0 \end{pmatrix} = i\mathbb{M}(y, \partial_t, k) - iA_0^0\,\mathbb{D}(\partial_t, k)\,, \\
&\mathbb{L}'' = i(k\partial_\theta)^2 \omega A_0^0\,.
\end{aligned} \tag{9.11}$$

Using the first identity yields

$$\Pi\ \mathbb{L}'\ \mathbb{Q}\ \mathbb{L}'\ \Pi \ = \ -\Pi\ \mathbb{M}\ \mathbb{Q}\ \mathbb{M}\ \Pi - \Pi\ A_0^0\ \mathbb{D}(\partial_t, k)\ \mathbb{Q}\ A_0^0\ \mathbb{D}(\partial_t, k)\ \Pi\,.$$

Applying (8.20) yields

$$\frac{1}{2}\Pi\ \mathbb{L}''\ \Pi \ = \ -\Pi\ \mathbb{M}\ \mathbb{Q}\ \mathbb{M}\ \Pi - \Pi\ A_0^0\ \mathbb{D}(\partial_t, k)\ \mathbb{Q}\ A_0^0\ \mathbb{D}(\partial_t, k)\ \Pi\,. \tag{9.12}$$

Equation (9.12) and the definition of $\mathbb{N}$ in (9.5), give

$$\begin{aligned}
&\Big(\Pi\,\mathbb{N}\,\Pi \ - \ \Pi\,\mathbb{M}\,Q\,\mathbb{M}\,\Pi\Big)\,w \\
&\quad = \Pi\Big[\partial_{\mathcal{T}} A_0^0 + i\omega A_0^1 + M + \frac{1}{2} i\ (k\partial_\theta)^2 \omega A_0^0 + A_0^0\,\mathbb{D}(\partial_t, k)\ \mathbb{Q}\ A_0^0\,\mathbb{D}(\partial_t, k)\Big]\Pi w\,.
\end{aligned} \tag{9.13}$$

Next replace k by ∂_x and simplify the right hand side of (9.13) using

$$\mathbb{D}(\partial_t, \partial_x)w = 0\,.$$

This yields

$$\Big(\Pi\,\mathbb{N}\,\Pi \ - \ \Pi\,\mathbb{M}\,Q\,\mathbb{M}\,\Pi\Big)\,w \ = \ \Pi\Big[\partial_{\mathcal{T}}A_0^0 + i\omega A_0^1 + M + \frac{1}{2}i\ \partial_\theta^2\omega(\partial_x, \partial_x)A_0^0\Big]\Pi w$$
$$= \ \Pi A_0^0\Pi\Big[\partial_{\mathcal{T}} + \frac{1}{2}i\ \partial_\theta^2\omega(\partial_x, \partial_x) + \gamma(t,x)\Big]w\,.$$

□

9.1. **Ray averages.** In general it is impossible to satisfy (9.9) exactly since all the terms are annihilated by $\partial_t + \mathcal{V}\partial_x$ except the $\gamma(t,x)$ term. If the coefficients are constant on group lines the γ term is too and one can construct infinitely accurate solutions (see [2]). In the present case we replace γ by its average on rays to find a solvable equation. Then estimate the error. For that estimate we impose an assumption on γ slightly stronger than the existence of ray averages. The material is recalled from [2] where the proofs and additional information can be found.

Assume that $\gamma \in C^\infty(\mathbb{R}^{1+3}\,;\,\mathrm{Hom}(\mathbb{K}))$ satisfies (8.21) and that the averages on rays exist as in (1.8). It follows that the $\widetilde{\gamma}$ is smooth and that the ray averages of the derivatives of γ exist uniformly on compacts and satisfy

$$\lim_{T\to+\infty}\ \Big\|\frac{1}{T}\int_0^T \partial_t^j\partial_x^\beta\gamma(t, x+\mathcal{V}t)\ dt\ -\ (-\mathcal{V}.\partial_x)^j\partial_x^\beta\widetilde{\gamma}(x)\Big\|_{L^\infty(\mathbb{R}^N)}\ =\ 0\,. \tag{9.14}$$

We need more than this. The function $\widetilde{\gamma}(x)$ is the average on the ray intersecting $t=0$ at x. The ray passing through the point (t,x) intersects $t=0$ at $x-\mathcal{V}t$. The function which assigns to (t,x) the average value of γ on the ray through (t,x) is equal to $\widetilde{\gamma}(x-\mathcal{V}t)$. The function that subtracts from $\gamma(t,x)$ its average on the group line through (t,x) is equal to $\gamma(t,x)-\widetilde{\gamma}(x-\mathcal{V}t)$.

Consider the $\mathrm{Hom}(\mathbb{K})$ valued solution g of the transport equation

$$\Big(\partial_t\ +\ \mathcal{V}.\partial_x\Big)g\ =\ \gamma(t,x)-\widetilde{\gamma}(x-\mathcal{V}t)\,,\qquad g\Big|_{t=0}=0\,. \tag{9.15}$$

Then

$$\frac{g(t,x)}{t}\ =\ \frac{1}{t}\int_0^t\gamma(s,\widetilde{x}+\mathcal{V}s)\,ds\ -\ \widetilde{\gamma}(\widetilde{x}),\qquad \widetilde{x} := x-\mathcal{V}t\,.$$

Assumption (1.8) is equivalent to the fact that this is $o(1)$ as $t\to+\infty$,

$$\lim_{t\to+\infty}\ \sup_{x\in\mathbb{R}^N}\ \frac{\|g(t,x)\|}{t}\ =\ 0\,. \tag{9.16}$$

Assume that γ satisfies the ray average hypothesis in Definition 1.8.

Example 9.4. i. *If $\gamma(t,x) = f(\ell(t,x))$ where $f(\theta)$ is a smooth periodic function of arbitrary period and ℓ is a linear functional then the ray average hypothesis is satisfied with $\beta = 0$.*

ii. *If $\mathcal{M} : \mathbb{R}^{1+N} \to \mathbb{R}^M$ is linear and satisfies the (generic) small divisor hypothesis*

$$\exists\, C > 0,\ m \in \mathbb{N}, \quad \forall n \in \mathbb{N}^M, \quad n.\mathcal{M}(1,\mathcal{V}) \neq 0 \ \Rightarrow\ |(n.\mathcal{M}(1,\mathcal{V})| \ \geq\ C\,|n|^{-m}\,,$$

then, for $h(\theta_1,\ldots,\theta_M) \in C^\infty(\mathbb{T}^M)$ the quasiperiodic function $\gamma(t,x) = h(\mathcal{M}(t,x))$ satisfies the hypothesis with $\beta = 0$ (see [17]*).*

iii. *Consider smooth almost periodic γ of the form*

$$\gamma(t,x) \ = \sum_{\eta \in \mathbb{R}^{1+N}} a_\eta\ e^{i\eta.(t,x)}\,, \tag{9.17}$$

where a_η vanish for all but a countable family of η and satisfy

$$\forall n \in \mathbb{N}, \qquad \sum_\eta \langle\eta\rangle^n\ \big|a_\eta\big| \ <\ \infty \qquad \langle\eta\rangle := (1+|\eta|^2)^{1/2}\,. \tag{9.18}$$

Then $\gamma(t,x) \ -\ \widetilde{\gamma}(x - \mathcal{V}t) \ = \ \sum_{\eta.(1,\mathcal{V})\neq 0} a_\eta\ e^{i\eta.(t,x)}$. Suppose that there is an $\alpha > 0$ so that for all n

$$\sum_{0<|\eta.(1,\mathcal{V})|<\delta} \langle\eta\rangle^n\ \big|a_\eta\big| \ =\ \mathcal{O}(\delta^\alpha), \qquad \delta \to 0\,. \tag{9.19}$$

Then the ray average hypothesis of Definition 1.8 holds with $\beta = \alpha/(\alpha+1)$.

9.2. **Using ray averages.** Rewrite (9.9) as

$$\Big(\partial_{\mathcal{T}} + \frac{1}{2} i\ \partial^2_\theta\omega(\partial_x,\partial_x) + \widetilde{\gamma}(x-\mathcal{V}t)\Big) w_0 \ =\ -\big(\partial_t + \mathcal{V}.\partial_x\big)\Pi w_1 - \big(\gamma(x) - \widetilde{\gamma}(x-\mathcal{V}t)\big) w_0\,. \tag{9.20}$$

This equation is satisfied by choosing w_0 and Πw_1 so that both sides are identically zero. The left yields equation (1.9). The initial value, $\widetilde{w}_0(0,x) \in \mathcal{S}(\mathbb{R}^3\,;\,\mathbb{K})$ is arbitrary.

Lemma 9.5. *For any $f \in \mathcal{S}(\mathbb{R}^3;\mathbb{K})$ there is a unique solution $\zeta(\mathcal{T},x) \in C^\infty(\mathbb{R}_{\mathcal{T}}\,;\,\mathcal{S}(\mathbb{R}^3;\mathbb{K}))$ to the Schrödinger initial value problem*

$$\Big(\partial_{\mathcal{T}} + \frac{1}{2} i\ \partial^2_\theta\omega(\partial_x,\partial_x) + \widetilde{\gamma}(x)\Big)\zeta \ =\ 0\,, \qquad \zeta(0) \ =\ f\,. \tag{9.21}$$

For each $m,s,r \in \mathbb{N}$, $|\alpha| \leq m$, $|\kappa| \leq s$ there exist constants $c(m,s,r), b(m,s)$ so that for all $\mathcal{T}$

$$\|x^\kappa\partial_x^\alpha\partial_{\mathcal{T}}^r\zeta(\mathcal{T})\|_{L^2(\mathbb{R}^3)} \leq c(m,s,r)e^{\left(1+b(m,s)\|\widetilde{\gamma}\|_{W^{m+s,\infty}}\right)\mathcal{T}}\ (1+\|\widetilde{\gamma}\|_{W^{r,\infty}})^r \sum_{|\ell|=0}^{s} \|x^\ell f\|_{H^{m+s-\ell+2r}}\,.$$

Proof. We present only the a priori estimates. Multiplying the equation by $\bar{\zeta}$ and taking the real part yields

$$\frac{1}{2}\frac{\partial}{\partial\mathcal{T}}\int_{\mathbb{R}^3} |\zeta|^2\ dx + \Re\int_{\mathbb{R}^3} \langle\widetilde{\gamma}\zeta,\zeta\rangle\ dx \ =\ 0\,.$$

Estimate

$$\Re \int_{\mathbb{R}^3} \langle \widetilde{\gamma}\zeta, \zeta\rangle \; dx \le \|\widetilde{\gamma}\|_{L^\infty} \int_{\mathbb{R}^3} |\zeta|^2 \; dx\,,$$

and integrate with respect to $\mathcal{T}$ to find

$$\frac{1}{2}\int_{\mathbb{R}^3} |\zeta(\mathcal{T})|^2 \; dx \le \frac{1}{2}\int_{\mathbb{R}^3} |f|^2 \; dx \;+\; \|\widetilde{\gamma}\|_{L^\infty} \int_0^{\mathcal{T}}\int_{\mathbb{R}^3} |\zeta(\mathcal{T})|^2 \; dx\,.$$

Gronwall's inequality gives

$$\|\zeta(\mathcal{T})\|_{L^2(\mathbb{R}^3)} \;\le\; e^{\|\widetilde{\gamma}\|_{L^\infty}\mathcal{T}}\; \|f\|_{L^2(\mathbb{R}^3)}\,. \tag{9.22}$$

The Duhamel relation

$$\begin{aligned}
\partial_{x_j}\zeta(\mathcal{T}) \;&=\; K(\mathcal{T},0)\partial_{x_j}\zeta(0) \;+\; \int_0^{\mathcal{T}} K(\mathcal{T},s)\; P(\partial_{x_j}\zeta)(s)\; ds \\
&=\; K(\mathcal{T},0)\partial_{x_j} f \;+\; \int_0^{\mathcal{T}} K(\mathcal{T},s)\; (-\partial_{x_j}\widetilde{\gamma})\zeta(s)\; ds
\end{aligned}$$

then yields the estimate

$$\|\partial_{x_j}\zeta(\mathcal{T})\|_{L^2(\mathbb{R}^3)} \;\le\; e^{\|\widetilde{\gamma}\|_{L^\infty}\mathcal{T}}\; \|\partial_{x_j} f\|_{L^2(\mathbb{R}^3)} \;+\; \int_0^{\mathcal{T}} e^{\|\widetilde{\gamma}\|_{L^\infty}(\mathcal{T}-s)}\|\nabla\widetilde{\gamma}\|_{L^\infty}\|\zeta(s)\|_{L^2(\mathbb{R}^3)}\, ds\,.$$

This combined with (9.22) gives

$$\|\zeta(\mathcal{T})\|_{H^1(\mathbb{R}^3)} \;\le\; 2\; e^{\|\widetilde{\gamma}\|_{L^\infty}\mathcal{T}}\; \|f\|_{H^1(\mathbb{R}^3)} \;+\; \int_0^{\mathcal{T}} e^{\|\widetilde{\gamma}\|_{L^\infty}(\mathcal{T}-s)}\|\nabla\widetilde{\gamma}\|_{L^\infty}\|\zeta(s)\|_{L^2(\mathbb{R}^3)}\, ds\,.$$

Now apply Gronwall's inequality to get

$$\|\zeta(\mathcal{T})\|_{H^1(\mathbb{R}^3)} \le 2e^{\|\widetilde{\gamma}\|_{W^{1,\infty}}\mathcal{T}}\; \|f\|_{H^1(\mathbb{R}^3)}\,.$$

By induction one proves that

$$\sup_{|\alpha|\le m} \big\|(\partial_x)^\alpha\,\zeta(\mathcal{T})\big\|_{L^2(\mathbb{R}^3)} \;\le\; 2\; e^{b(m)\,\|\widetilde{\gamma}\|_{W^{m,\infty}}\mathcal{T}}\; \|f\|_{H^m(\mathbb{R}^3)}\,.$$

Let us prove the weighted estimate $x^\kappa\partial_x^\alpha$. The commutator of x_j with the Schrödinger operator S in (9.21) is the first order scalar differential operator

$$[S,x_j] = i\sum_l (\partial_\theta^2\omega)_{lj}\partial_{x_j}\,.$$

Therefore, for $|\alpha|\le m$

$$\begin{aligned}
x_j\partial_x^\alpha\zeta(\mathcal{T}) \;&=\; K(\mathcal{T},0)x_j\partial_x^\alpha\zeta(0) \;+\; \int_0^t K(\mathcal{T},s)\; S(x_j\partial_x^\alpha\zeta)(s)\; ds \\
&=\; K(\mathcal{T},0)x_j\partial_x^\alpha f \;+\; \int_0^t K(\mathcal{T},s)\; [S,x_j](\partial_x^\alpha\zeta)(s)\; ds\,,
\end{aligned}$$

which yields

$$\|x_j\partial_x^\alpha\zeta(\mathcal{T})\|_{L^2(\mathbb{R}^3)} \le c\, e^{\left(1+b(m+1)\|\widetilde{\gamma}\|_{W^{m+1,\infty}}\right)\mathcal{T}}\; \Big[\|x_j g\|_{H^m(\mathbb{R}^3)} \;+\; \|f\|_{H^{m+1}(\mathbb{R}^3)}\Big]\,.$$

By induction one proves, for $|\kappa| \leq s$,

$$\|x^\kappa \partial_x^\alpha \zeta(\mathcal{T})\|_{L^2(\mathbb{R}^3)} \leq c(m,s) e^{\left(1+b(m,s)\|\widetilde{\gamma}\|_{W^{m+s,\infty}}\right)\mathcal{T}} \sum_{|\ell|=0}^{s} \|x^\ell f\|_{H^{m+s-\ell}} \,. \tag{9.23}$$

The time derivative commutes with the Schrödinger operator, therefore, for each $r \in \mathbb{N}$, $\partial_{\mathcal{T}}^r \zeta$ satisfies the same equation as ζ with initial condition

$$|\partial_{\mathcal{T}}^r \zeta(0)| \leq C(r)(1 + \|\widetilde{\gamma}\|_{W^{r,\infty}})^r \sum_{j \leq 2r} |\partial_{\mathcal{T}}^j f| \,.$$

Apply (9.23) to $\partial_{\mathcal{T}}^r \zeta$ to find

$$\|x^\kappa \partial_x^\alpha \partial_{\mathcal{T}}^r \zeta(\mathcal{T})\|_{L^2(\mathbb{R}^3)} \leq c(m,s,r) e^{\left(1+b(m,s)\|\widetilde{\gamma}\|_{W^{m+s,\infty}}\right)\mathcal{T}} (1 + \|\widetilde{\gamma}\|_{W^{r,\infty}})^r \sum_{|\ell|=0}^{s} \|x^\ell f\|_{H^{m+s-\ell+2r}} \,.$$

$\square$

Given $\widetilde{w}_0(0) = f \in \mathcal{S}(\mathbb{R}^3; \mathbb{K})$ choose $\widetilde{w}_0$ the solution provided by Lemma 9.5. It satisfies for all α,

$$(x\,, \partial_{\mathcal{T},t,x})^\alpha w_0 \ \in\ L^\infty([0,T]_{\mathcal{T}} \times \mathbb{R}_t \times \mathbb{R}^3_x\,;\, \mathbb{K})\,. \tag{9.24}$$

Setting the right hand side of (9.20) equal to zero yields an equation that is solved using a $\mathrm{Hom}(\mathbb{K})$ valued integrating factor $g(t,x)$,

$$\Pi w_1 = g(t,x)\, w_0, \tag{9.25}$$

where g is the solution of (9.15). The ray average hypothesis with parameter $0 \leq \beta < 1$ yields estimates for the derivatives of g and therefore those of Πw_1,

$$\langle t \rangle^{-\beta}\ (x\,, \partial_{t,x})^\alpha (\Pi w_1) \ \in\ L^\infty([0,T] \times \mathbb{R}_t \times \mathbb{R}^3_x\,;\, \mathbb{K})\,.$$

The component $(I - \Pi)w_1$ is given by (9.6) in terms of w_0 so (9.24) implies,

$$(x\,, \partial_{\mathcal{T},t,x,y})^\alpha (I - \Pi) w_1 \ \in\ L^\infty([0,T] \times \mathbb{R}_t \times \mathbb{R}^3_x \times \mathbb{R}^3_y)\,,$$

with w_1 is periodic in y. This completes the determination of w_0 and w_1. At this stage one has $r_{-1} = r_0 = \Pi\, r_1 = 0$. We choose w_2 to that $(I - \Pi) r_1 = 0$. The latter equation holds if and only if

$$(I - \Pi) w_2 \ =\ \mathbb{Q}\,\mathbb{M} w_1 + \mathbb{Q}\,\mathbb{N} w_0\,. \tag{9.26}$$

This determines $(I - \Pi)w_2$. On the other hand, Πw_2 does not affect the profiles r_{-1}, r_0, r_1. It is chosen equal to zero,

$$\Pi\, w_2 \ =\ 0\,. \tag{9.27}$$

The estimates for w_0, w_1 imply that the y-periodic w_2 satisfies estimates analogous to those of w_1 so,

$$\langle t\rangle^{-\beta}\ (x\,,\partial_{\mathcal{T},t,x,y})^\alpha w_j\ \in\ L^\infty([0,T]\times\mathbb{R}_t\times\mathbb{R}^3_x\times\mathbb{R}^3_y),\qquad j=1,2\,. \tag{9.28}$$

This completes the determination of the profiles so that

$$r_{-1}=r_0=r_1=0\,. \tag{9.29}$$

Theorem 9.6. *If $f\in\mathcal{S}(\mathbb{R}^3;\mathbb{K})$ there is a unique $w_0\in C^\infty(\mathbb{R};\mathcal{S}(\mathbb{R}^3;\mathbb{K}))$ satisfying* (1.9) *with $w_0(0)=f$. Define w_1, w_2 and v^h by* (9.6), (9.25), (9.26), (9.27) *and* (9.1) *respectively. If u^h is the exact solution of $P^h u^h=0$ with $u^h\big|_{t=0}=v^h\big|_{t=0}$, then for all α*

$$\sup_{t\in[0,T/h]}\ \big\|(x\,,\,h\,\partial_{t,x})^\alpha(u^h-v^h)\big\|_{L^2(\mathbb{R}^3)}\ \le\ C(\alpha)\,h^{1-\beta}\quad 0<h<1\,. \tag{9.30}$$

Proof. Let $m\in\mathbb{N}$. The bound (9.28) and the identity (9.29) yield the residual estimate

$$\big\|\langle t\rangle^{-\beta}(x\,,\,h\,\partial_{t,x})^\alpha\big(P^h v^h\big)\big\|_{L^\infty([0,T/h]\times\mathbb{R}^3)}\ \le\ C(\alpha)\,h^2,\qquad h\in]0,1[\,. \tag{9.31}$$

This combined with Theorem 4.1 and Remark 4.3 shows that there exists a constant $C(m,T)$ such that for all $t\in[0,T/h]$

$$\begin{aligned}\big\|(u^h-v^h)(t)\big\|_{m,h}\ &\le\ C(m,T)\Big(\big\|(u^h-v^h)(0)\big\|_{m,h}\ +\ \int_0^{T/h}\big\|P^h\,v^h(s)\big\|_{m,h}\,ds\Big)\\ &=\ C(m,T,\beta)\Big(\big\|(u^h-v^h)(0)\big\|_{m,h}\ +\ \mathcal{O}\big(h^{1-\beta}\big)\Big)\,.\end{aligned} \tag{9.32}$$

For the first term in the rhs of (9.32) we follow the proof of Theorem 8.6 which yields in this case

$$\big\|(u^h-v^h)(0)\big\|_{m,h}\le\mathcal{O}\big(h^{1-\beta}\big)\,.$$

The error estimate with polynomial weights x^α requires an additional weighted stability estimate. We will use the following notation. For a function $u(t,x)$, integers $\ell,m\ge 0$, define

$$\|u(t)\|_{\ell,m,h}\ :=\ \sum_{|\beta|\le\ell\,,\,|\alpha|\le m}\big\|x^\beta(h\partial_{t,x})^\alpha u(t)\big\|_{L^2(\mathbb{R}^3)}\,.$$

Lemma 9.7. *Under the assumptions of Theorem 4.1, there exist constants $c(\ell,m)$, $C(m,h)$ such that*

$$\|u(t)\|_{\ell,m,h}\ \le\ c(\ell,m)\,e^{c(\ell,m)\,C(m,h)\,t}\ \|u(0)\|_{\ell,m,h}\,. \tag{9.33}$$

Proof. It is a commutator argument resembling the proof of Theorem 4.1. The proof is inductive in ℓ. The case $\ell=0$ is provided by Theorem 4.1. Assume that (9.33) holds for $\ell=\bar\ell$. The case $\bar\ell+1$ is proved by applying the inductive hypothesis to $x_j u$ for $j=1,2,3$ and using the commutator relation

$$[P^h,x_j]=A_j\qquad j=1,2,3\,.$$

An application of Gronwall's inequality finishes the proof of the lemma. □

Using the weighted estimates as above proves the theorem. □

Proof of Theorem 1.9 Write

$$\underline{u}^h - \underline{v}^h \;=\; (u^h - v^h) \;+\; (\underline{u}^h - u^h) \;+\; (v^h - \underline{v}^h)\,.$$

The preceding theorem estimates the first term. It suffices to show that the differences $u^h - \underline{u}^h$ and $v^h - \underline{v}^h$ have similar upper bounds. The first follows from the stability for Maxwell's equations proved in Lemma 9.7. The second follows from Lemma 9.5. □

References

[1] G. Allaire, M. Palombaro, J. Rauch, *Diffractive behavior of the wave equation in periodic media: weak convergence analysis,* Annali di Matematica Pura e Applicata **188**, 561-590, (2009).

[2] G. Allaire, M. Palombaro, J. Rauch, *Diffractive geometric optics for Bloch waves*, Arch. Ration. Mech. Anal., **202**, 373-426, (2011)

[3] G. Allaire, M. Palombaro, J. Rauch, *A bound on group velocity for Bloch wave packets*, preprint (2012).

[4] G. Allaire, A. Piatnitski, *Homogenization of the Schrödinger equation and effective mass theorems,* Comm. Math Phys. 258, pp.1-22 (2005).

[5] H. Altug, J. Vuckovic, *Experimental demonstration of the slow group velocity of light in two-dimensional coupled photonic crystal microcavity arrays,* Appl. Phys. Lett. **86**, 111102-1 to 111102-3 (2004).

[6] M. Bajcsy, A. Zibrov, M. Lukin, *Stationary pulses of light in an atomic medium,* Nature **426** (2003).

[7] K. Barrailh, D. Lannes, *A general framework for diffractive optics and its applications to lasers with large spectrum and short pulses*, SIAM, Journal on Mathematical Analysis **34**, no. 3, 636-674, (2003)

[8] A. Bensoussan, J.-L. Lions, G. Papanicolaou, *Asymptotic Analysis for Periodic Structures,* North-Holland, Amsterdam (1978).

[9] F. Bloch, *Uber die quantenmechanik der electronen in kristallgittern,* Z. Phys. **52**, pp. 555–600, (1928).

[10] L. Brillouin, *Propagation of Waves in Periodic Structures*, Dover, New York (1953).

[11] R. Courant, *Methods of Mathematic Physics vol. II*, 2nd. ed. Springer-Verlag (1962).

[12] P. Donnat, J.-L. Joly, G. Métivier, J. Rauch, *Diffractive nonlinear geometric optics,* Séminaire sur les Equations aux Dérivées Partielles, 1995–1996, Exp. No. XVII, 25 pp., Ecole Polytechnique, Palaiseau (1996).

[13] G. Floquet, *Sur les équations différentielles linéaires à coefficients périodiques*, Ann. Sci. École Norm. Sup. Sér. 2, 12 (1883), pp. 47–89.

[14] H. Gersen, T. Karle, R. Engelen, W. Bogaerts, J. Korterik, N. van Hulst, T. Krauss, and L. Kuipers, *Real-space observation of ultraslow light in photonic crystal waveguides,* Phys. Rev. Lett. 94(2005) 073903-1-073903-4.

[15] W.R. Hamilton, *Third supplement to an essay on the theory of systems of rays*, Trans. Roy. Irish Acad. **17**, 1-144, (1833).

[16] L.V. Hau, S. E. Harris, Z. Dutton, C. Behroozi, *Light speed reduction to 17 meters per second in an ultracold atomic gas,* Nature **397**, 594-598, (1999).

[17] J.-L. Joly, G. Métiver, and J. Rauch, *Generic rigorous asymptotic expansions for weakly nonlinear multidimensional oscillatory waves,* Duke Math J., **70**, 373-404, (1993).

[18] J.-L. Joly, G. Métiver, and J. Rauch, *Diffractive nonlinear geometric optics with rectification*, Indiana U. Math. J. **47**, 1167-1242, (1998).

[19] J.L. Joly, G. Métivier and J. Rauch, *Hyperbolic domains of dependence and Hamilton-Jacobi equations*, J. Hyp. Diff. Eq., **2**(2005)713-744.

[20] T. Kato, *Perturbation Theory for Linear Operators,* Springer-Verlag, Berlin (1966).

[21] P. Kuchment, *The mathematics of photonic crystals,* Mathematical modeling in optical science, 207–272, Frontiers Appl. Math., 22, SIAM, Philadelphia (2001).

[22] P. D. Lax, *Asymptotic solutions of oscillatory initial value problems*, Duke Math. J. **24**, 627-646, (1957).

[23] J. Leray, *Lectures on Hyperbolic Partial Differential Equations*, Institute for Advanced Study (1953).

[24] T. Nishida and J. Rauch, The initial boundary value problem for the inviscid compressible Euler equations, unpublished manuscript.

[25] J. Rauch, *Hyperbolic Partial Differential Equations and Geometric Optics*, American Mathematical Society (2012).

[26] M. Reed, B. Simon, *Methods of Modern Mathematical Physics*, Academic Press, New York (1978).

[27] P. St. J. Russell, *Photonic crystal fibers*, J. Lightwave. Technol., 24 (12), 4729-4749, (2006).

[28] D. Sjöberg, Ch. Engström, G. Kristensson, D. Wall, N. Wellander, A Floquet-Bloch decomposition of Maxwell's equations applied to homogenization, SIAM MMS, 4, 149-171 (2005).

[29] Y.A. Vlasov, S. Petit, G. Klein, B. Hönerlage, and C. Hirlmann, *Femtosecond measurements of the time of flight of photons in a three-dimensional photonic crystal*, Phys. Rev. E **60**, 1030-1035, (1999).

[30] C. Wilcox, *Theory of Bloch waves*, J. Analyse Math. **33**, 146–167, (1978)